Precise Explicit Approximations of the Colebrook-White Equation for Engineering Systems

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Abstract. Modern automated engineering systems have variable hydraulic/aerodynamic conditions with Reynolds number from zero to hundred thousand with a wide range of roughness. Simple approximations of the Colebrook-White equation cannot give enough precision. The aim of the work is a universal simple precise approximation of the Colebrook-White equation. The methods are selected by the analysis of the equation curve. In the whole range of turbulent flow, it is near to linear. Thus, Newton's method is very effective. The algorithm is proposed for getting high-precision approximations. The results are two simple explicit ones for rough and careful calculations with a deviation of 5.36 % and 0.00072 % in a wide range of parameters. It is shown on the examples of the objects: the highest building "Biotecton" and researches of "green roofs" in a wind tunnel. The scientific novelty is that we scientifically grounded the effective usage of Newton's method, which provides new universal, precise and simple explicit approximations of Colebrook-White equation. The practical value is that the approximations are covered different practical tasks of hydraulic and aerodynamic calculations in the whole range of turbulent flow.

Keywords: Colebrook-White equation, Hydraulic calculation, Aerodynamic Calculation, Microclimate System, Turbulent Flow, Approximation.

1 Introduction

Quality and reliability of engineering systems in buildings depends on the correctness of hydraulic and aerodynamic calculations of pipelines or air ducts. Friction losses in most cases are significant in the total pressure losses. Most flows in the systems are turbulent, and the friction losses correspond to the accurate but implicit Colebrook-White equation [1]:

$$EQ = f^{-1/2} + \frac{2}{\ln(10)}\ln\left(\frac{2.51}{Re}f^{-1/2} + \frac{\Delta_e/D}{3.71}\right) = \frac{2}{\ln(10)}\ln\left(\frac{a}{3.71Re}\right) + f^{-1/2} = 0, \qquad (1)$$

where Δ_e – equivalent roughness of the pipeline, m; Re – Reynolds number; D – hydraulic (equivalent) diameter of the pipeline, m; a - parameter:

$$a = Re(\Delta_e/D) + 9.3121/f^{-1/2}. \quad (2)$$

In older systems with an approximately constant hydraulic or aerodynamic regime, most flows have a developed turbulent regime, and the Reynolds number is more than 10,000. In these conditions very simple logarithmic (after conversion to natural one) and power approximations by A. Altshul [2] were used:

$$f^{-1/2} = -\left(1.8/\ln(10)\right)\ln\left(\left((\Delta_e/D)/10\right) + 7/Re\right); \quad (3)$$

$$f = 0.11\left((\Delta_e/D) + (68/Re)\right)^{1/4}. \quad (4)$$

Due to increasing energy efficiency, modern systems are highly automated. A significant amount of research is devoted to the appropriate air distributors [3, 4, 5, 6] or optimal control strategies [7, 8, 9]. However, the simulation of hydraulic and aerodynamic conditions [10] is no less important. The Reynolds number varies from zero to hundreds of thousands. The range of equivalent roughness, m, in modern pipelines and air ducts has also expanded. Plastic technologies decrease it by order. Flexible corrugated pipelines rise it up to commensurable to the diameter [m]. Therefore, universal, simple and accurate explicit approximations of the Colebrook-White equation are becoming especially relevant today. Due to the forced use of iterative procedures, the modelling of the variable thermal-hydraulic regime of a renovated one-pipe heating system [10] of an apartment building took more than a day. Through the efforts of the English Wikipedia Society, a table of historically significant approximations was created, which at May 2020 had 26 positions [11]. At May 2020, the last entry corresponds to 2018 [12]. Dejan Brkić and Pavel Prax [1] obtained the most accurate modern approximations in 2019 by one and two steps of Padé approximation. It has 0.0259 % of deviation, but it is too bulky. If we pick a simple fast-converging method, a compact approximation will be found. For example, Newton's method has been effectively used for computer calculation of Darcy coefficient [13, 14].

2 Aim

The aim of the work is a simple precise approximation of the Colebrook-White equation, which is applicable for manual calculation and complex simulations.

3 Method

To choose a method for solving equation (1), let us analyse its properties. Partial derivatives of the solution according to the parameters $Re \geq 2320$ and $\Delta_e/D \geq 0$ (to save time taken in the Maxima system) $\partial(f^{-1/2}) / \partial Re > 0$ and $\partial(f^{-1/2}) / \partial(\Delta_e / D) < 0$. Therefore, the solution $f^{-1/2}$ monotonically increases with increasing Reynolds number and decreases with increasing relative roughness Δ_e / D. The range of the solution is determined from Eq. (1). For $Re = \infty$ and $\Delta_e/D = 0$ ($f = 0$). At $Re = 2320$ and $\Delta_e / D = 1$ Eq. (1) has been solved numerically. Thus

$$\left(f^{-1/2}\right)_{min} = 1.1348004636910905664 \le f^{-1/2} \le \infty \quad (5)$$

$$0 \le f \le 0.77653492097452793220 = f_{max}. \quad (6)$$

$$a_{min} = 9.3121 \cdot 1.1348004636910905664 = 10.567375397937803782 < a < \infty. \quad (7)$$

The first derivative of the EQ function of Eq. (1) by the unknown parameter $f^{-1/2}$ monotonically decreases with increasing $f^{-1/2}$ and $Re\ \Delta_e/D$:

$$1 < \partial \mathrm{EQ}/\partial(f^{-1/2}) = 1 + \left((18.6242/\ln(10))/a\right) < 1.7654111816109958780. \quad (8)$$

The narrow range of change of the derivative (8) indicates the closeness of the equation function EQ to the linear one. The second derivative is always negative and changes from minus 0.67448966236883048814 to zero. The function is convex with the curvature

$$\kappa = \left(173.43041282/\ln(10)\right)/\left(\left(\left(a + \left(18.6242/\ln(10)\right)\right)/a^{1/3}\right)^2 + a^{4/3}\right)^{3/2}. \quad (9)$$

The derivative of (9) by a has one positive root $a_{extr} = 2.328025\ (17^{1/2} - 1) / \ln(10) \approx$ $\approx 3.1576109808930112860$, at which $\kappa_{extr} = 4\ (17^{1/2} - 1) \ln(10) / (3\ (17^{1/2} + 7))^{3/2} \approx$ $\approx 0.14922492819316431408$. For $a \to \infty$, the curvature (9) goes to zero. Therefore, the root corresponds to the maximum curvature (8). In the range (5), the curvature monotonically decreases: $\kappa_{max} = 0.080752349164383260460 \le \kappa \le 0$. Therefore, the curve should be close enough to the line.

Let us test the maximum nonlinear curve EQ. We will assume that all approximations in the process of solving Eq. (1) are close enough to the root (otherwise, these parameters in the process of solving should be considered as an independent). As it is shown above, the solution $f^{-1/2}$ increases by Re and decreases by Δ_e / D. To clarify the total effect on the curvature, let us express the parameter a from Eq. (1):

$$a = 3.71\ Re/10^{(1/2)\ f^{-1/2}}. \quad (10)$$

The parameter a by Eq. (10) increases with Re and decreases with $f^{-1/2}$, which, in turn, decreases (as it is shown above) with $\Delta e / D$. Therefore, taking into account the decrease in curvature (9) by a, the largest value of κ corresponds to $Re = 2320$ and $\Delta_e / D = 0$ (Fig. 2). The obtained result indicates the practical linearity of the curve EQ by the solution $f^{-1/2}$. Thus, the methods of linear approximation are the most suitable for solving the Eq. (1). Preference should be given to a method that requires only one approximation, i.e. the Newton method. At manual calculations, the first approximation can be taken not less than $(f^{-1/2})_{min}$.

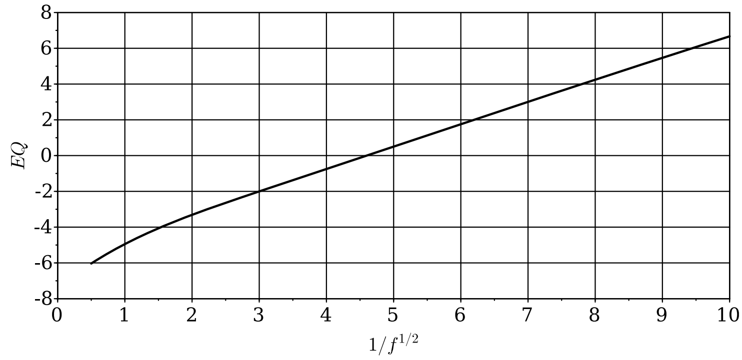

Fig. 1. The graph of the function EQ in Eq. (1) depending on the approximation of $f^{-1/2}$

In machine calculation, excessive verification reduces the performance of the algorithm. However, Fig. 2 in the critical case of the greatest curvature shows that the proximity to linearity is maintained when the approximation deviates more than two times in the smaller direction.

According to Eqs. (1) and (8), the formula of Newton's method is simple and can be used for computer and manual calculations:

$$f_{i+1}^{-1/2} = f_i^{-1/2} - \frac{EQ\left(f_i^{-1/2}\right)}{\left(dEQ/d\left(f^{-1/2}\right)\right)_{f_i^{-1/2}}} = \frac{\frac{18,6242}{\ln(10)} f_i^{-1/2} - \frac{2}{\ln(10)} a_i \ln\left(\frac{a_i}{3,71Re}\right)}{(18,6242/\ln(10)) + a_i}, \tag{11}$$

where i is the iteration number, and $i = 0$ corresponds to the first approximation. For the efficient calculation of *Eq.* (11), it is recommended to calculate the constants with the maximum accuracy of the corresponding computer system (or calculator) and enter or store them digitally. Very high accuracy is not required for engineering calculations. Therefore, instead of performing iterations (11), another approach is proposed to achieve high accuracy. We assume a sufficiently dense grid of Reynolds numbers $Re = 2320...10^9$ with a variable step ΔRe: $\Delta Re = 20$ at $Re \leq 10000$, $\Delta Re = 200$ at $10000 < Re \leq 100000$, $\Delta Re = 2000$ at $100000 < Re \leq 1000000$, and so on. Similarly, we take a grid of relative equivalent roughness $\Delta_e / D = 0...0.1$ with a variable step $\Delta(\Delta_e / D)$: $\Delta(\Delta_e / D) = 2 \cdot 10^{-8}$ for $\Delta_e / D \leq 10^{-5}$, $\Delta(\Delta_e / D) = 2 \cdot 10^{-7}$ for $10^{-5} < \Delta_e / D \leq 10^{-4}$, $\Delta(\Delta_e / D) = 2 \cdot 10^{-6}$ for $10^{-4} < \Delta_e / D \leq 10^{-3}$, etc. We accept the first approximation, in this paper – (3). To increase accuracy, all numerical coefficients are considered as unknowns C_1, C_2 and C_3:

$$f^{-1/2} = -C_1 \ln\left(\left((\Delta_e / D) / C_2\right) + \left(C_3 / Re\right)\right). \tag{12}$$

Their values in Eq. (3) are considered as the first approximation. In the system of computer algebra (in this work – SciLab) a program has been created that single time numerically solves Eq. (1) with the maximum possible accuracy in each node of the grid. After that, a function multiple times returns the maximum relative deviation on the grid of the value according to Eq. (12) from the obtained solution of Eq. (1) for any values of the coefficients C_1, C_2 and C_3. The obtained result is optimized (in this work by the fsolve function according to the Nelder-Mead method) to minimize the maximum deviation. During the process, accuracy increases by an order in comparison with Eq. (3). Next, the grid should be extended to test the possibility of expanding the range of application of the formula without reducing accuracy.

After that, a single iteration by Eqs. (2) and (11) is applied to the first approximation (12). In this case, again, all numerical coefficients are replaced by unknowns C_j. The obtained coefficients C_1, C_2 and C_3 are also not considered as constants but are used for the first approximation for optimization, which is performed similarly. This allows significant refining of the result obtained by Newton's method.

Similarly, two iterations of Newton's methods with more unknown coefficients can be performed. Since Newton's iterative process was broken at the first iteration because of optimization, there is no guarantee that the previously obtained coefficients C_j are a better second approximation than the standard coefficients in Eqs. (2) and (11). Thus, standard coefficients were used to ensure the convergence of the process.

4 Results

As a result, after a few days of machine calculations we have (equations are given in the order of calculation in a form that provides a minimum of operations):

- with a deviation up to 5.36 % (Fig. 2) within $Re = 2320...10^9$, $\Delta_e / D = 0...0.65$ for rough engineering calculations:

$$f = \left(0.8284 \ln\left(\frac{\Delta_e/D}{4.913} + \frac{10{,}31}{Re}\right)\right)^{-2} \qquad (13)$$

- with a deviation up to 0.00072 % (Fig. 3) at $Re = 2320...10^9$, $\Delta_e / D = 0...0.65$ for the most application in engineering and science (further refinement is impractical):

$$\begin{cases} f_0^{-1/2} = -0.79638 \ln\left(\frac{\Delta_e/D}{8.208} + \frac{7.3357}{Re}\right), \\ a_1 = (Re\Delta_e/D) + 9{,}3120665 f_0^{-1/2}, \\ f = \left(\frac{8.128943 + a_1}{8.128943 f_0^{-1/2} - 0.86859209 a_1 \ln(a_1/3.7099535 Re)}\right)^2. \end{cases} \qquad (14)$$

Large Reynolds numbers (over 10^6) are used in special technologies and constructions. An example is the project of the highest skyscraper "Biotecton" [15] of 1 km high for cities with polluted air. Its ventilation takes very clean air at the level of 1 km and supplies it to through an air-duct of 10 m in diameter. It is thermally insulated and soundproofed. Only technical and economic indicators limit the air velocity. The Reynolds number exceeds 10 million. The equations (14) are the best universal ones for calculations of all microclimate systems in the object.

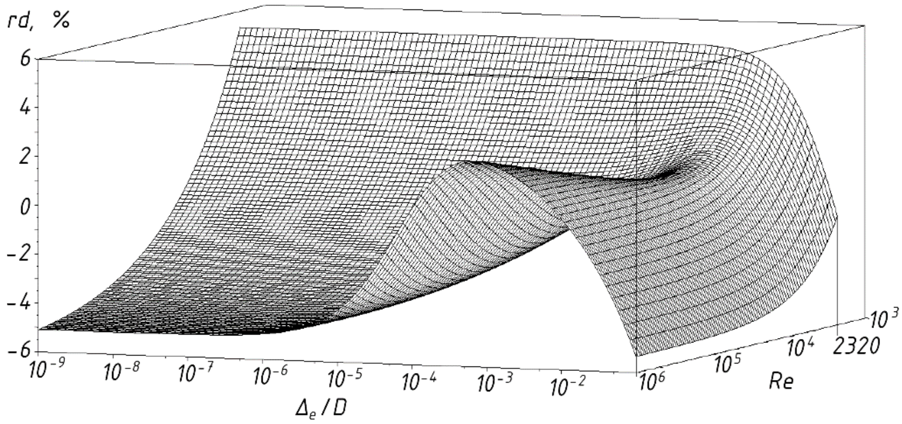

Fig. 2. Deviation of *rd*,%, approximation (13) from the Colebrook-White Eq. (1) depending on the Reynolds number *Re* and the relative equivalent roughness Δ_e / D.

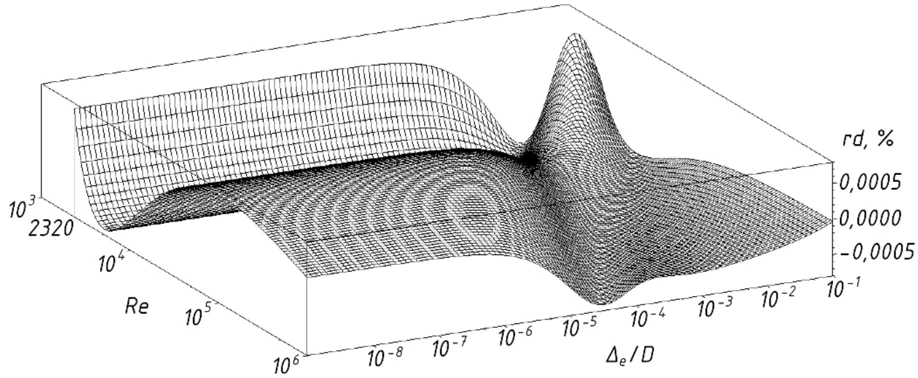

Fig. 3. Deviation of *rd*,%, approximation (14) from the Colebrook-White Eq. (1) depending on the Reynolds number *Re* and the relative equivalent roughness Δ_e / D.

An example of pipelines with high equivalent roughness is the study of "Green structures" in a straight wind tunnel. The first author's research was carried out [16] in the Eiffel chamber (Fig. 4 a) with unlimited internal height [m]. To increase the length of the models it is planned to use a wind tunnel (Fig. 4 b) with a straight duct of a cross-section of 1 × 1 m (hydraulic diameter D_h = 1 m) and a flow velocity u_p to u = 10 m/s (Re = 666667). The grass on the "green roof" model grows up to (averaged) 450 mm (Fig. 4 c). Such a model with a width of 1000 mm is planned to be installed at the bottom of the duct. The equivalent roughness of other walls can be considered 0.1 mm [17]. The average equivalent channel roughness will be (450 + 0.1 + 0.1 + 0.1) / 4 = 112.6 mm. The relative equivalent roughness is Δ_e / D = 0.1126. According to the Eq. (14) f = 0.1085. For comparison, by Eq. (4) f = 0.06373, i.e. 1.7 times less. Using air density $\rho = 1.2$ kg/m^3 (at temperature of T = 293.15 K), pressure loss per meter according to Darcy's formula [2] $\Delta p_\ell = (f / D_h)(\rho u^2 / 2) = 6.51$ Pa/m.

a b c

Fig. 4. Wind tunnels: a – Eiffel chamber; b - straight wind tunnel at Czestochowa Polytechnic (Czestochowa, Poland); c – the model with grass of 300...500 mm high.

This pressure gradient at a model length of 1...2 m is insignificant and it is within the pressure gradient that occurs when the wind goes around the roofs of real buildings.

5 Scientific novelty and practical significance

The scientific novelty is that we scientifically grounded the effective usage of Newton's method, which provides new universal, precise and simple explicit approximations of Colebrook-White equation. The practical value is that the approximations are covered different practical tasks of hydraulic and aerodynamic calculations, and allow simulation of variable hydraulic conditions in microclimate systems with adequate computational resources.

Conclusions

The obtained approximations of the solution of the Colebrook-White equation allow solving effectively a wide range of problems – engineering calculations and scientific researches. The effectiveness of the use of approximations is confirmed by the example of studies of green roofs in the wind tunnel in the form of a straight channel.

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