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D. O. Chernyshev, V. A. Glyva, S. I. Shamanskyi

FLUID DYNAMICS

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Автори:
Д. О. Чернишев – д. т. н., проф., перший проректор КНУБА;
В. А. Глива – д. т. н., проф., зав. кафедри фізики КНУБА;
С. Й. Шаманський – д. т. н., доц., проф. кафедри водопостачання та водовідведення КНУБА.

Рецензенти:

М. М. Біляєв – д. т. н., проф., завідувач кафедри гідравліки, водопостачання та фізики Українського державного університету науки і технологій.

В. М. Бадах – к. т. н., с. н. с., завідувач кафедри гідрогазових систем Національного авіаційного університету

С. В. Селичко – к. т. н., доц., доцент кафедри водопостачання та водовідведення Київського національного університету будівництва і архітектури.

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У навчальному посібнику висвітлюються загальні відомості про основні характеристики крапельних рідин та газів, закони рівноваги, кінематики та динаміки рідин. Розглядаються також процеси, що відбуваються у пригранничому шарі при обтіканні поверхонь різних форм та при різних режимах руху. Аналізуються закономірності стискуваного руху газів. Розглядаються методи аналізу розмірностей та основи теорії подібності. Розглядаються також не традиційні для подібних посібників теми. А саме: транспортна теорема Рейнольдса, методи аналізу контрольного об’єму для розв’язування задач гідрогазодинаміки.

Для здобувачів галузі знань 14 «Енергетична інженерія» спеціальності 144 «Теплоенергетика» першого (бакалаврського) рівня вищої освіти.

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CONTENTS

INTRODUCTION

Fluid dynamics is an important part of the professional course training of energy workers. Knowledge of the laws of movement and equilibrium of liquids and gases is necessary for understanding technologies used in the energy industry.

Fluid dynamics is the study of fluids in motion. But before studying a fluid in motion, it is necessary to familiarize yourself with fluids at rest. Both gases and liquids can be classified as fluids. The number of fluid engineering applications is very wide.

This tutorial considers the subject as an applied technical science that based on the laws of statics, kinematics, and dynamics of the working environment (liquid and gas) forms methods of using theoretical provisions for solving engineering problems in various industries. The development of scientific and technological progress constantly needs expansion of the use of sophisticated devices and automated technological equipment (metal-cutting machines; forge-press; stamping and bending machines; industrial manipulators; flexible production lines, etc.).

The purpose of teaching Fluid dynamics is to give the students opportunities to master the skills of using laws of statics and dynamics for the design of hydraulic and pneumatic devices and automatic machines.

Before starting to study the discipline, it requires assimilation of the following disciplines: Mathematics (precalculus and calculus); Physics; Analytical mechanics.

THEME 1. SOME CHARACTERISTICS OF FLUIDS

Fluid is defined as a substance that deforms continuously (flows) when acted on by a shearing stress of any magnitude.

A ***shearing stress*** (force per unit area) is created whenever a tangential force acts on a surface as shown in Fig. 1.1.

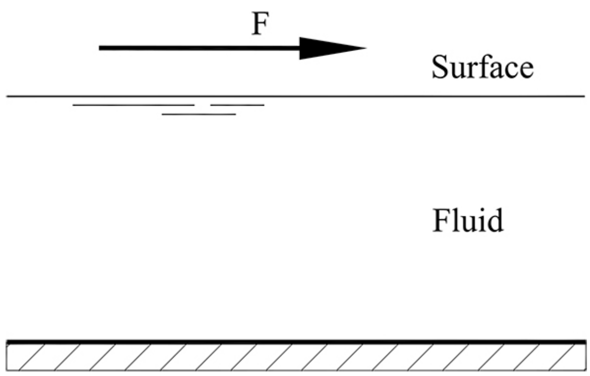


Fig. 1.1. Creating a shearing stress by a tangential force

When common solids such as steel or other metals are acted on by a shearing stress, they will initially deform (usually a very small deformation), but they will not continuously deform.

Some materials, such as slurries, tar, toothpaste, and so on, are not easily classified since they will behave as a solid if the applied shearing stress is small, but if the stress exceeds some critical value, the substance will flow.

Thus, all the fluids we will be concerned with in this text will conform to the definition of fluid given previously.

Continuum

For gases at normal pressures and temperatures, the spacing between molecules is on the order of 10^{-6} mm. For liquids, it is on the order of 10^{-7} mm. The number of molecules per cubic millimeter is on the order of 10^{18} mm for gases, and 10^{21} mm for liquids. It means that the number of molecules in a very tiny volume is huge. So, the idea of using average values of fluid characteristics, taken over this volume, is reasonable. We thus assume that all the fluid characteristics we are interested in (pressure, velocity, etc.) vary continuously throughout the fluid – that is, we treat the fluid as a ***continuum***. This concept will be valid for all the circumstances considered in this text. (One area of fluid mechanics for which the continuum concept breaks down is in the study of rarefied gases such as would be encountered at very high altitudes. In this case, the spacing between air molecules can become large and the continuum concept is no longer acceptable).

The qualitative description of fluid characteristics is conveniently given in terms of certain primary quantities, such as length, L, time, T, mass, M, and temperature, Θ. These primary quantities can then be used to provide a qualitative description of any other secondary quantity: for example, area L^2, velocity LT^{-1}, density ML^{-3}, and so on.

Dimensions

The primary quantities are also referred to as ***basic dimensions***. For a wide variety of problems involving fluid mechanics, only the three basic dimensions, L, T, and M are required. Alternatively, L, T, and F could be used, where F is the basic dimension of force. Since Newton's law states that force is equal to mass times acceleration, it follows that $F = MLT^{-2}$, or $M = FL^{-1}T^2$. Thus, secondary quantities expressed in terms of M can be expressed in terms of F. For example, stress, σ, is a force per unit area, so that $\sigma = FL^{-2}$, but an equivalent dimensional equation is $\sigma = ML^{-1}T^{-2}$.

Table 1 provides a list of dimensions for a number of common physical quantities.

All theoretically derived equations are ***dimensionally homogeneous*** – that is, the dimensions of the left side of the equation must be the same as those on the right side, and all additive separate terms must have the same dimensions.

Equations that are restricted to a particular system of units can be denoted as ***restricted homogeneous equations***, as opposed to equations valid in any system of units, which are ***general homogeneous equations***.

International System (SI)

In SI the unit of length is the meter (m), the time unit is the second (s), the mass unit is the kilogram (kg), and the temperature unit is the kelvin ($°K$)

$$°K = °C + 273.15$$

The force unit called the newton (N)

Table 1.1. A list of dimensions for several common physical quantities

Quantity	Dimensions	
	MLTϴ	**FLTϴ**
Length	L	L
Area	L^2	L^2
Volume	L^3	L^3
Velocity	LT^{-1}	LT^{-1}
Acceleration	LT^{-2}	LT^{-2}
Speed of sound	LT^{-1}	LT^{-1}
Volume flow	L^3T^{-1}	L^3T^{-1}
Mass flow	MT^{-1}	FTL^{-1}
Pressure, stress	$ML^{-1}T^{-2}$	FL^{-2}
Stain rate	T^{-1}	T^{-1}
Angle	None	None
Angular velocity	T^{-1}	T^{-1}
Viscosity	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	L^2T^{-1}	L^2T^{-1}
Surface tension	MT^{-2}	FL^{-1}
Force	MLT^{-2}	F
Torque	ML^2T^{-2}	FL
Power	ML^2T^{-3}	FLT^{-1}
Work, energy	ML^2T^{-2}	FL
Density	ML^{-3}	FT^2L^{-4}
Temperature	Θ	Θ
Specific heat	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Expansion coefficient	Θ^{-1}	Θ^{-1}

$$1\ N = (1\ kg)\cdot(1\ m/s^2).$$

Standard gravity in SI is $g = 9.807\ m/s^2$ (commonly approximated as $g = 9.81\ m/s^2$).

The unit of work in SI is the joule (*J*),

$$1\ J = 1\ N{\cdot}m.$$

The unit of power is the watt (*W*),

$$1\ W = 1\ J/s = 1\ N{\cdot}m/s.$$

British Gravitational (BG) System

In the BG system, the unit of length is the foot (*ft*), the time unit is the second (*s*), the force unit is the pound (*lb*), and the temperature unit is the degree Fahrenheit (°*F*) or the absolute temperature unit is the degree Rankine (°*R*),

$$°R = °F + 549.67$$

The mass unit called the slug,

$$1\ lb = (1\ slug){\cdot}(1\ ft/s^2).$$

Earth's standard gravity is taken as g = 32.174 ft/s^2 (commonly approximated as g = 32.2 ft/s^2).

The relative sizes of the SI and BG units of length, mass, and force are shown in Fig. 1.2.

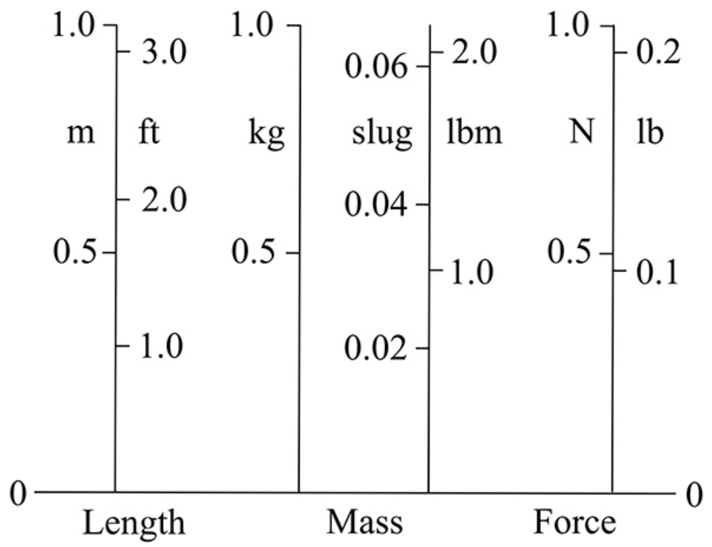


Fig. 1.2. The relative sizes of the SI and BG units of length, mass, and force

Tables 1.2 and 1.3 provide conversion factors for some quantities that are commonly encountered in fluid mechanics.

Table 1.2 Conversion Factors from BG Units to SI Units

	To convert from	**to**	**Multiply by**
Acceleration	ft/s^2	m/s^2	3.048 E – 1
Area	ft^2	m^2	9.290 E – 2
Density	lbm/ft^3	kg/m^3	1.602 E + 1
	$slugs/ft^3$	kg/m^3	5.154 E + 2
Energy	Btu	J	1.055 E + 3
	ft·lb	J	1.356
Force	lb	N	4.448
Length	ft	m	3.048 E – 1
	in	m	2.540 E – 2
	mile	m	1.609 E + 3
Mass	lbm	kg	4.536 E – 1
	slug	kg	1.459 E + 1
Power	ft·lb/s	W	1.356
	hp	W	7.457 E + 2
Pressure	in. Hg (60 ^{0}F)	N/m^2	3.377 E + 3
	lb/ft^2 (psf)	N/m^2	4.788 E + 1
	lb/in^2 (psi)	N/m^2	6.895 E + 3
Specific weight	lb/ft^3	N/m^3	1.571 E + 2
Temperature	^{0}F	^{0}C	$T_C = (5/9) \cdot (T_F - 32^0)$
	^{0}R	^{0}K	5.556 E – 1
Velocity	ft/s	m/s	3.048 E – 1
	mi/hr (mph)	m/s	4.470 E – 1
Viscosity (dynamic)	$lb \cdot s/ft^2$	$N \cdot s/m^2$	4.788 E + 1
Viscosity (kinematic)	ft^2/s	m^2/s	9.290 E – 2
Volume flowrate	ft^3/s	m^3/s	2.832 E – 2
	gal/min (gpm)	m^3/s	6.309 E – 5

Table 1.3 Conversion Factors from SI Units to BG Units

	To convert from	**to**	**Multiply by**
Acceleration	m/s^2	ft/s^2	3.281
Area	m^2	ft^2	1.076 E + 2
Density	kg/m^3	lbm/ft^3	6.243 E – 2
	kg/m^3	$slugs/ft^3$	1.940 E – 3
Energy	J	Btu	9.478 E – 4
	J	ft·lb	7.346 E – 1
Force	N	lb	2.248 E – 1
Length	m	ft	3.281
	m	in	3.937 E + 1
	m	mile	6.214 E – 4
Mass	kg	lbm	2.205
	kg	slug	6.852 E – 2
Power	W	ft·lb/s	7.376 E – 1
	W	hp	1.341 E – 3
Pressure	N/m^2	in. Hg (60 ^0F)	2.961 E – 4
	N/m^2	lb/ft^2 (psf)	2.089 E – 2
	N/m^2	lb/in^2 (psi)	1.450 E – 4
Specific weight	N/m^3	lb/ft^3	6.366 E – 3
Temperature	^0C	^0F	$T_F = 1.8\ T_C + 32^0$
	^0K	^0R	1.800
Velocity	m/s	ft/s	3.281
	m/s	mi/hr (mph)	2.237
Viscosity (dynamic)	$N \cdot s/m^2$	$lb \cdot s/ft^2$	2.089 E – 2
Viscosity (kinematic)	m^2/s	ft^2/s	1.076 E + 1
Volume flowrate	m^3/s	ft^3/s	3.531 E + 1
	m^3/s	gal/min (gpm)	1.585 E + 4

The broad subject of ***fluid mechanics*** can be generally subdivided into ***fluid statics***, in which the fluid is at rest, and ***fluid dynamics***, in which the fluid is moving.

Density

The ***density*** ρ of a fluid is defined as its mass per unit volume. In the BG system, ρ has units of *slugs/ft*^{3}, and in SI the units are *kg/m*^{3}.

The small change in the density of water with large variations in temperature is illustrated in Fig. 1.3.

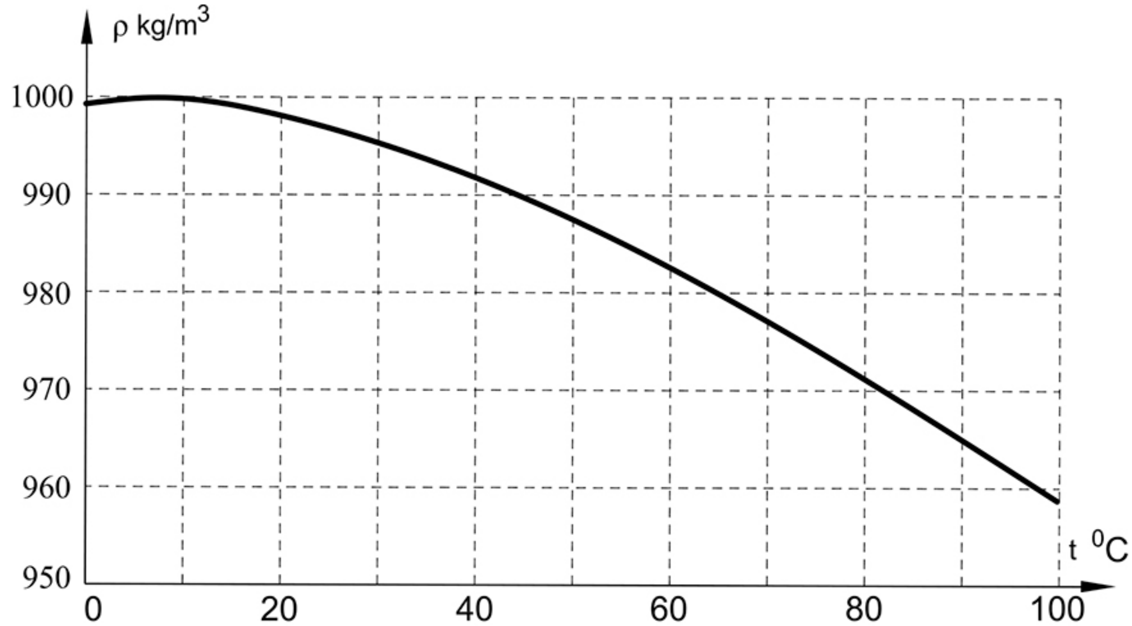


Fig. 1.3. Change in the density of water with variations in temperature

Specific weight

The ***specific weight*** of a fluid is defined as its weight per unit volume. Thus, specific weight is related to density through the equation

$$\gamma = \rho \cdot g \tag{1.1}$$

In the BG system, γ has units of *lb/ft*^{3} and in SI the units are *N/m*^{3}.

The ***specific gravity*** of a fluid, designated as **SG**, is defined as the ratio of the density of the fluid to the density of water at some specified temperature. Usually, the specified temperature is taken as 4 *°C* (39.2 *°F*). The value of SG does not depend on the system of units used.

Ideal gas law

Gases are highly compressible in comparison to liquids. Changes in gas density are directly related to changes in pressure and temperature through the equation

$$\rho = \frac{p}{R \cdot T} , \qquad (1.2)$$

where p, absolute pressure, ρ, density, T, absolute temperature, and R, gas constant. This equation is commonly termed the ***ideal*** or ***perfect gas law***, or the ***equation of state for an ideal gas***.

Pressure has the dimension of FL^{-2} and in BG units expressed as lb/ft^2 (*psf*) or lb/in^2 (*psi*) and in SI units as N/m^2. In SI, 1 N/m^2 is defined as a ***pascal***, abbreviated as *Pa*.

Standard sea-level atmospheric pressure (by international agreement) is 14.696 *psi* (*abs*) or 101.33 *kPa* (*abs*). For most calculations, these pressures can be rounded to 14.7 *psi* and 101 *kPa*, respectively.

The gas constant R depends on the particular gas and is related to the molecular weight of the gas.

Viscosity

To determine the viscosity of a fluid, consider a hypothetical experiment in which a material is placed between two very wide parallel plates as shown in Fig. 1.4.

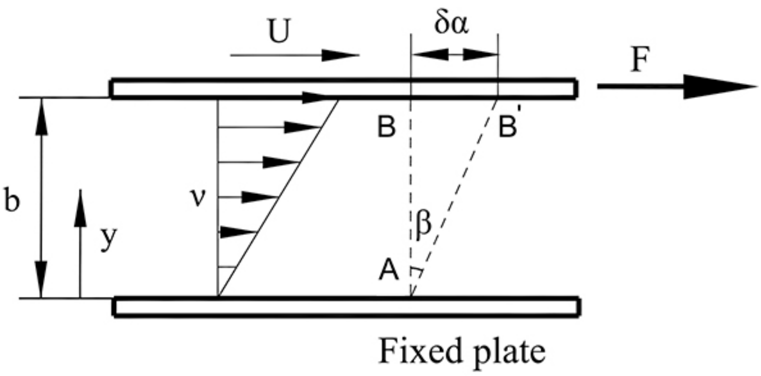


Fig. 1.4. Behavior of a fluid placed between two parallel plates

The bottom plate is rigidly fixed, but the upper plate is free to move. When the force *P* is applied to the upper plate, it will move continuously with a velocity, *U* (after the initial transient motion has died out). The fluid in contact with the upper plate moves with the plate velocity, *U*, and the fluid in contact with the bottom fixed plate has a zero velocity. Thus, a velocity gradient, *du/dy*, is developed in the fluid between the plates. In this particular case, the velocity gradient is a constant since $du/dy = U/b$, but in more complex flow situations, this is not true.

The result indicates that for common fluids such as water, oil, gasoline, and air the shearing stress and rate of shearing strain (velocity gradient) can be related with a relationship of the form

$$\tau = \mu \frac{du}{dy}, \tag{1.3}$$

where the constant of proportionality is called the absolute viscosity, dynamic viscosity, or simply the viscosity of the fluid. Plots of τ versus *du/dy* should be linear with the slope equal to the viscosity as illustrated in Fig. 1.5.

Fluids for which the shearing stress is linearly related to the rate of shearing strain (also referred to as the rate of angular deformation) are designated as Newtonian fluids.

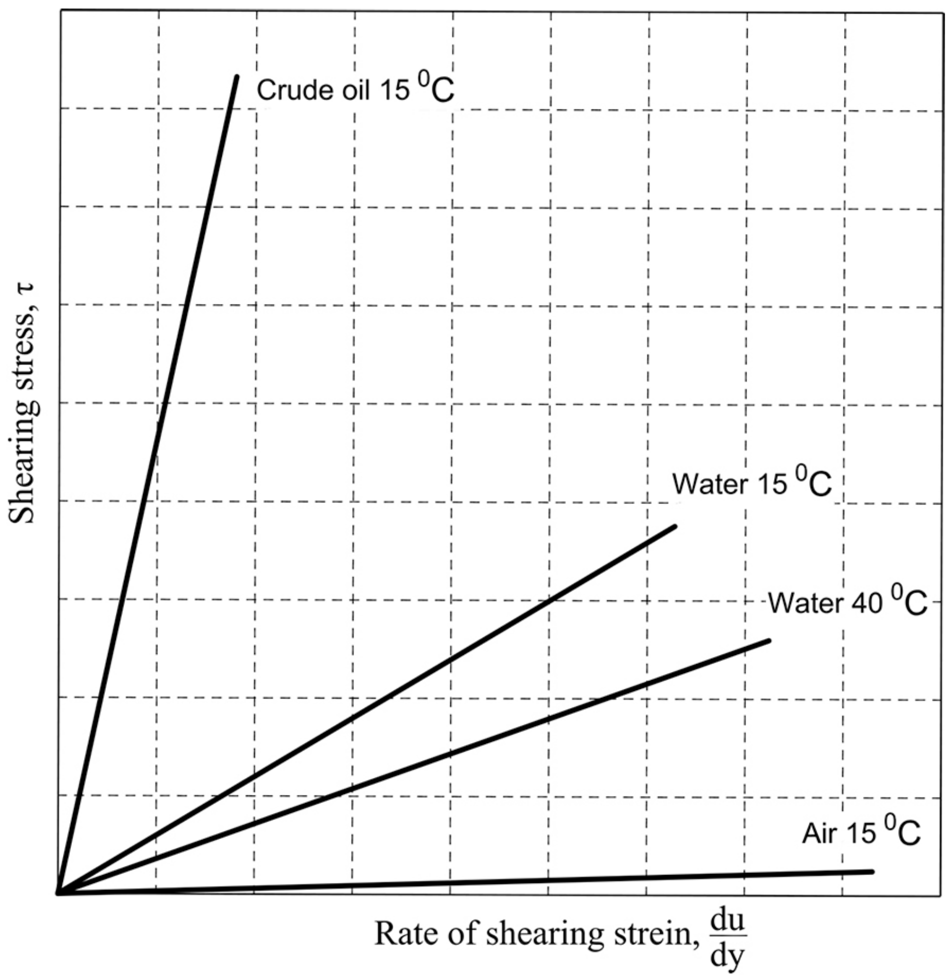


Fig. 1.5. Variation of shearing stress with rate of shearing strain for common fluids

Fluids for which the shearing stress is not linearly related to the rate of shearing strain are designated as non-Newtonian fluids.

From the previous equation, it can be readily deduced that the dimensions of viscosity are FTL^{-2}. Thus, in BG units, viscosity is given as $lb{\cdot}s/ft^2$ and in SI units as $N{\cdot}s/m^2$.

Quite often viscosity appears in fluid flow problems combined with the density in the form

$$\nu = \frac{\mu}{\rho} \ , \qquad (1.4)$$

This ratio is called ***kinematic viscosity***. The dimensions of kinematic viscosity are L^2/T and the BG units are ft^2/s and SI units are m^2/s.

Dynamic viscosity is often expressed in the metric CGS (centimeter-gram-second) system with units of $dyne{\cdot}s/cm^2$. This combination is called a ***poise***, abbreviated *P*. In the CGS system, kinematic viscosity has units of cm^2/s and this combination is called a ***stoke***, abbreviated *St*.

Compressibility

A property, which is commonly used to characterize compressibility, is the ***bulk modulus*** E_v, defined as

$$E_v = -\frac{dp}{dW/W} \ , \qquad (1.5)$$

where *dp* is the differential change in pressure needed to create a differential change in volume *dW*, of a volume *W*. The negative sign is included since an increase in pressure will cause a decrease in volume. Since a decrease in volume of a given mass, $m = \rho W$, will result in an increase in density, the ***bulk modulus*** can also be expressed as

$$E_v = \frac{dp}{dp/\rho} \ , \qquad (1.6)$$

The bulk modulus has dimensions of pressure. Large values for the bulk modulus indicate that the fluid is relatively incompressible. We conclude that liquids can be considered incompressible for most practical engineering applications.

When gases are compressed (or expanded), the relationship between pressure and density depends on the nature of the process. If the compression or expansion takes place under constant temperature conditions (isothermal process), then from the perfect gas law

$$\frac{p}{\rho} = constant \ .$$

If the compression or expansion is frictionless and no heat is exchanged with the surroundings (isentropic process), then

$$\frac{p}{\rho^k} = constant \ ,$$

where k is the ratio of the specific heat at constant pressure, c_p, to the specific heat at constant volume, c_v ($k = c_p/c_v$). The two specific heats are related to the gas constant, R, through the equation $R = c_p - c_v$.

The bulk modulus for gases can be determined by obtaining the derivative *dp/dρ* from the equations above and substituting the results into the bulk modulus equation.

It follows that for an isothermal process

$$E_v = p \ ,$$

and for an isentropic process

$$E_v = kp \ .$$

In both cases, the bulk modulus varies directly with pressure.

Speed of sound

The speed of sound is related to changes in the pressure and density of the fluid medium through the equation

$$c = \sqrt{\frac{dp}{d\rho}} \ ,$$

or in terms of the bulk modulus

$$c = \sqrt{\frac{E_v}{\rho}} \ .$$

There is negligible heat transfer during sound propagation and the process is assumed to be isentropic, so that

$$c = \sqrt{\frac{kp}{\rho}} \ ,$$

and making use of the ideal gas law, it follows that

$$c = \sqrt{kRT} \ . \qquad (1.7)$$

Thus, for ideal gases, the speed of sound is proportional to the square root of the absolute temperature.

Vapor pressure

When an equilibrium condition is reached during the evaporation process so that the number of molecules leaving the surface is equal to the number entering, the vapor is said to be saturated and the pressure that the vapor exerts on the liquid surface is termed the ***vapor pressure***, p_v.

Boiling, which is the formation of vapor bubbles within a fluid mass, is initiated when the absolute pressure in the fluid reaches the vapor pressure.

In flowing fluids, it is possible to develop very low pressure due to the fluid motion, and if the pressure is lowered to the vapor pressure, boiling will occur. When vapor bubbles are formed in a flowing fluid, they have usually swept along into regions of higher pressure where they suddenly collapse with sufficient intensity to actually cause structural damage.

The formation and subsequent collapse of vapor bubbles in a flowing fluid is called cavitation. It is an important fluid flow phenomenon.

Surface tension

The intensity of the molecular attraction per unit length along any line in the surface of a liquid is called ***surface tension***. For a given liquid the surface tension σ depends on temperature as well as the other fluid it is in contact with at the interface. The dimensions of surface tension are FL^{-1} with BG units of *lb/ft* and SI units of *N/m*.

There are capillary actions in small tubes, which involves a liquid-gas-solid interface, is caused by surface tension. The case illustrated in Fig. 1.6a is an attraction (adhesion) between the wall of the tube and liquid molecules which is strong enough to overcome the mutual attraction (cohesion) of the molecules and pull them up the wall. Hence, the liquid is said to wet the solid surface.

The height, h, is governed by the value of the surface tension, σ, the tube radius, R, the specific weight of the liquid, γ, and the angle of contact, Θ, between the fluid and tube. From the free-body diagram of Fig. 1.6b we see that the vertical force due to the surface tension is equal to $2\pi R\ \sigma \cdot \cos\ \Theta$ and the weight is $\gamma\ \pi R^2 h$ and these two forces must balance for equilibrium. Thus,

$$2\pi R\,\sigma \cdot \cos\theta = \gamma\,\pi R2h\ ,$$

so that the height is given by the relationship

$$h = \frac{2\sigma \cdot \cos\theta}{\gamma\, R}\ . \qquad (1.8)$$

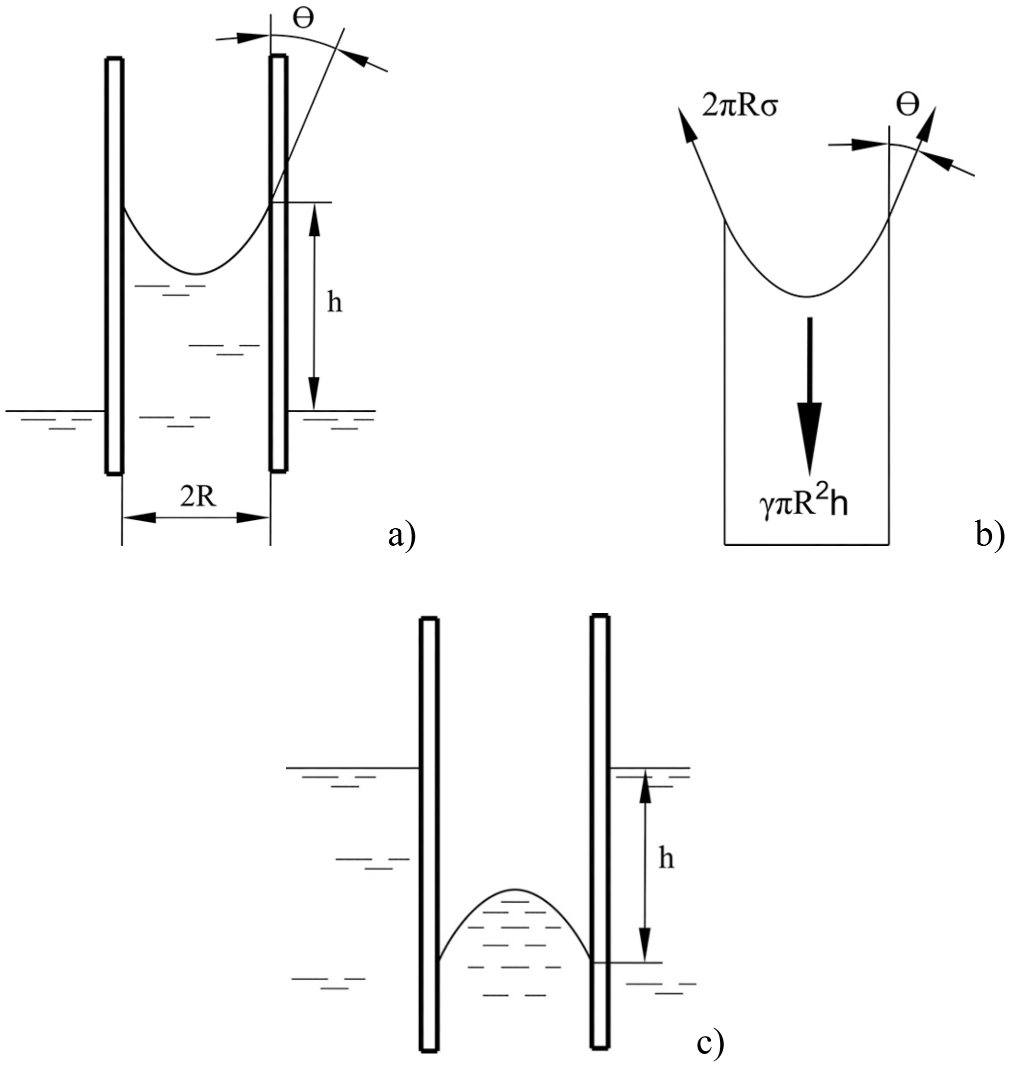


Fig. 1.6. Effect of capillary action in small tubes: (a) rise of column for a liquid that wets the tube; (b) free-body diagram for calculating column height; (c) depression of column for a nonwetting liquid.

If adhesion of molecules to the solid surface is weak compared to the cohesion between molecules, the liquid will not wet the surface and the level in a tube placed in a nonwetting liquid will actually be depressed, as shown in Fig. 1.6c.

THEME 2. PASCAL'S LAW

Pascal's law says the pressure at a point in a fluid at rest, or in motion, is independent of direction as long as there are no shearing stresses present.

Consider the free-body diagram, illustrated in Fig. 2.1, that was obtained by removing a small triangular wedge of fluid from some arbitrary location within a fluid mass. Since we are considering the situation in which there are no shearing stresses, the only external forces acting on the wedge are the pressure and the weight. The assumption of zero shearing stresses will still be valid as long as the fluid element is at rest or moves as a rigid body; that is, there is no relative motion between adjacent elements.

For simplicity, the forces in the *x* direction are not shown.

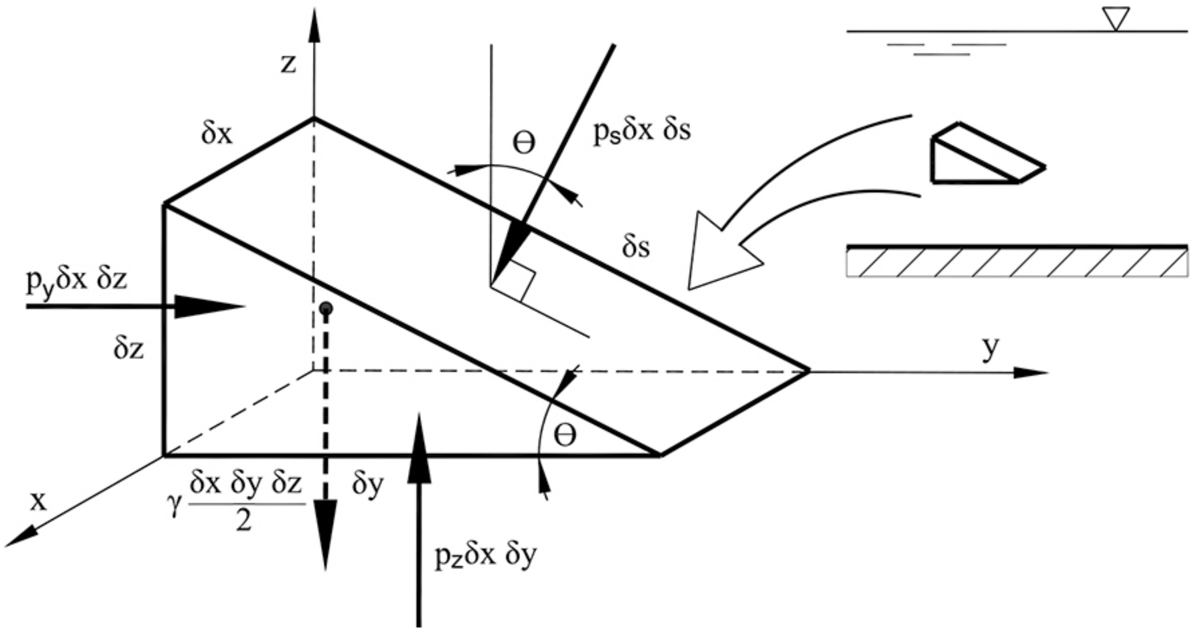


Fig. 2.1. Forces on an arbitrary wedge-shaped element of fluid.

The sum of all forces acting on the element of fluid along each axis must be equal zero. Consider *y* axes

$$\sum F_y = 0\,. \qquad (2.1)$$

It means that

$$P_y \cdot (\delta z) \cdot (\delta x) - P_s(sin\theta) \cdot (\delta s) \cdot (\delta x) = 0$$

where P_y, pressure along positive *y* direction, $(\delta z) \cdot (\delta x)$, area upon which P_y acts, $P_s(sin\Theta)$, component of P_s pressure along negative *y* direction, $(\delta s) \cdot (\delta x)$, area upon which P_s acts.

It follows from the geometry that

$$\delta z = \delta s \cdot sin\theta$$
$$\delta y = \delta s \cdot cos\theta$$

So that the equations of forces along *y* axes can be rewritten as

$$P_y sin\theta \cdot (\delta s) \cdot (\delta x) - P_s sin\theta \cdot (\delta s) \cdot (\delta x) = 0,$$

or

$$P_y sin\theta \cdot \delta s \cdot (\delta x) = P_s sin\theta \cdot \delta s \cdot (\delta x).$$

It follows that

$$P_y = P_s.$$

The same thing we can do along *x* axes and prove that $P_x = P_s$.
Now consider *z* axes

$$\sum F_z = 0\,.$$

It means that

$$P_z \cdot (\delta y) \cdot (\delta x) - P_s(cos\theta) \cdot (\delta s) \cdot (\delta x) - \gamma \frac{(\delta y) \cdot (\delta s) \cdot (\delta x)}{2} = 0$$

where $\gamma \frac{(\delta y)\cdot(\delta s)\cdot(\delta x)}{2}$, weight of the fluid element.

Considering the geometry, the equations of forces along *z* axes can be rewritten as

$$P_z \cdot (\delta s) \cdot (cos\theta) \cdot (\delta x) - P_s(cos\theta) \cdot (\delta s) \cdot (\delta x) - \gamma \frac{(\delta y) \cdot (\delta s) \cdot (\delta x)}{2} = 0,$$

or

$$P_z - P_s = \gamma \frac{(\delta y) \cdot (\delta s) \cdot (\delta x)}{2(cos\theta) \cdot (\delta s) \cdot (\delta x)} = \frac{(\delta y)}{2(cos\theta)}.$$

Since we are really interested in what is happening at a point, we take the limit as δy approaches zero (while maintaining the angle Θ), and it follows that

$$P_z - P_s = 0.$$

Now we can state

$$P_x = P_y = P_z = P_s, \tag{2.2}$$

which means that the pressure here is independent of direction.

Pressure at a point

Now a different approach can be taken.

The important question is how does the pressure in a fluid, in which there are no shearing stresses, vary from point to point.

Consider a small rectangular element of fluid removed from some arbitrary position within the mass of fluid of interest as illustrated in Fig. 2.2. There are two types of forces acting on this element: surface forces due to the pressure and a body force equal to the weight of the element.

If we let the pressure at the center of the element be designated as *p*, then the average pressure on the various faces can be expressed in terms of *p* and its derivatives, as shown in Fig. 2.1.

The resultant surface force in the *y* direction is

$$\sum F_y = \left[p - \frac{\partial p}{\partial y}\cdot\frac{\delta y}{2}\right]\cdot(\delta x)\cdot(\delta z) - \left[p + \frac{\partial p}{\partial y}\cdot\frac{\delta y}{2}\right]\cdot(\delta x)\cdot(\delta z) = 0\,,$$

or

$$p(\delta x)\cdot(\delta z) - \frac{\partial p}{\partial y}\cdot\frac{\delta y}{2}\cdot(\delta x)\cdot(\delta z) - p(\delta x)\cdot(\delta z) - \frac{\partial p}{\partial y}\cdot\frac{\delta y}{2}\cdot(\delta x)\cdot(\delta z) = 0,$$

finally

$$-\frac{\partial p}{\partial y}\cdot(\delta x)\cdot(\delta z)\cdot(\delta z) = 0\,.$$

It follows that

$$\frac{\partial p}{\partial y} = 0\,.$$

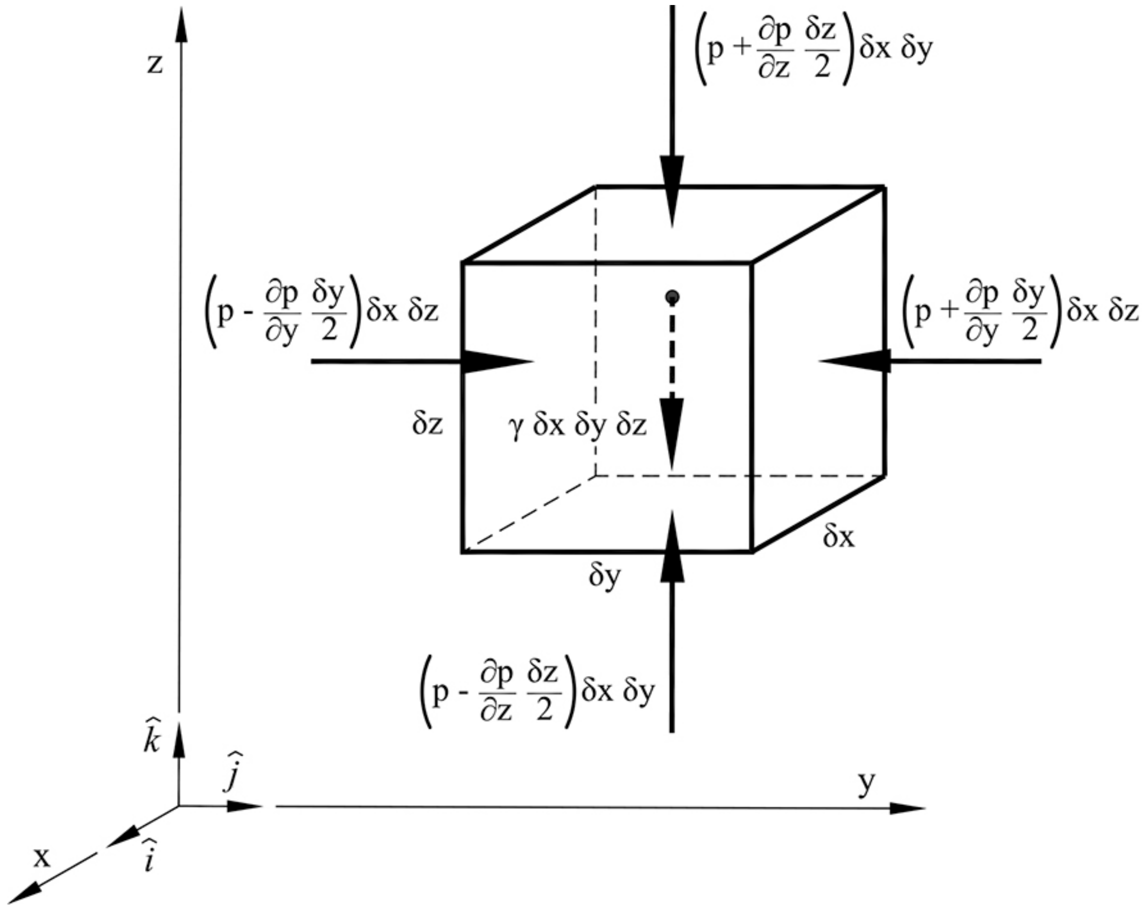


Fig. 2.2. Surface and body forces acting on small fluid element.

The same thing we can do in the *x* direction and prove that

$$\frac{\partial p}{\partial x} = 0 .$$

So, the conclusion is: ***the pressure does not vary on horizontal planes***.

This rule is trough for fluid element. If we consider large amount of fluid the rule holds under the next conditions: fluid is still, fluid is the same and continuous from point to point.

The resultant surface force in the *z* direction is

$$\sum F_z = \left[p - \frac{\partial p}{\partial z} \cdot \frac{\delta z}{2}\right] \cdot (\delta x) \cdot (\delta y) - \left[p + \frac{\partial p}{\partial z} \cdot \frac{\delta z}{2}\right] \cdot (\delta x) \cdot (\delta y) -$$
$$-\gamma \cdot (\delta x) \cdot (\delta x) \cdot (\delta y) = 0 ,$$

or

$$p(\delta x)\cdot(\delta y)-\frac{\partial p}{\partial z}\cdot\frac{\delta z}{2}\cdot(\delta x)\cdot(\delta y)-p(\delta x)\cdot(\delta y)-\frac{\partial p}{\partial z}\cdot\frac{\delta z}{2}\cdot(\delta x)\cdot(\delta y)$$

$$-$$

$$-\gamma\cdot(\delta x)\cdot(\delta x)\cdot(\delta y)=0,$$

finally

$$-\frac{\partial p}{\partial z}-\gamma=0\,.$$

It follows that

$$\frac{\partial p}{\partial z}=-\gamma\,. \qquad (2.3)$$

The conclusion is: ***the pressure changes in downward direction***.

Equation (2.3) is the ***fundamental equation for fluids at rest*** and can be used to determine how pressure changes with elevation. Minus sign in the equation indicates that the pressure gradient in the vertical direction is negative; that is, the pressure decreases as we move upward in a fluid at rest.

Since the specific weight is equal to the product of fluid density and acceleration of gravity ($\gamma = \rho g$) changes in γ are caused by a change in either ρ or g. For most engineering applications the variation in g is negligible, so our main concern is with the possible variation in the fluid density. In general, a fluid with constant density is called an ***incompressible fluid***.

Equation (2.3) can be directly integrated

$$\int_{p_1}^{p_2} d\rho=-\gamma\int_{z_1}^{z_2} dz\,.$$

or

$$p_2-p_1=-\gamma(z_2-z_1)\,.$$

finally

$$p_1-p_2=\gamma(z_2-z_1)\,. \qquad (2.4)$$

where p_1 and p_2 are pressures at the vertical elevations z_1 and z_2.

Equation (2.4) can be written in the compact form

$$p_1-p_2=\gamma h\,. \qquad (2.5)$$

or

$$p_1 = \gamma h + p_2 \,. \tag{2.6}$$

where h is the distance, $z_2 - z_1$, which is the depth of fluid measured downward from the location of p_2. This type of pressure distribution is commonly called a ***hydrostatic distribution***.

Equation (2.6) shows that in an incompressible fluid at rest the pressure varies linearly with depth.

The pressure difference between two points can be specified by the distance h since

$$h = \frac{p_2 - p_1}{\gamma} \,.$$

In this case h is called the ***pressure head*** and is interpreted as the height of a column of fluid of specific weight γ required to give a pressure difference $p_1 - p_2$.

When a liquid has a free surface, it is convenient to use this surface as a reference plane. The ***reference pressure*** p_0 would correspond to the pressure acting on the free surface (which would frequently be atmospheric pressure), and thus it follows that the pressure p at any depth h below the free surface is given by the equation:

$$p = \gamma h + p_0 \,. \tag{2.7}$$

Consider a reservoir of complicated form which contains some fluids illustrated in Fig. 2.3. The reservoir has closed and open parts filled with fluid to different levels (*f.s.* means free surface). The fluids in the reservoir have different density (different specific weight $\gamma_1 \neq \gamma_2 \neq \gamma_3$).

Points *a*, *b*, *c*, *d* and *e* they all lie along the same horizontal line. Despite this, the pressure will not be the same at all points

$$P_a = P_b = P_c \neq P_d \neq P_e \,.$$

Inequalities are because fluids are not the same at points *c* and *d*, fluid is not continuous between points *c* and *e*.

Points *j*, *i*, *h*, *g* and *f* also lie along the same horizontal line. Despite this, the pressure will also not be the same at all points

$$P_j = P_i = P_h = P_g \neq P_f$$

Inequality is because fluid is not continuous between points *g* and *f*.

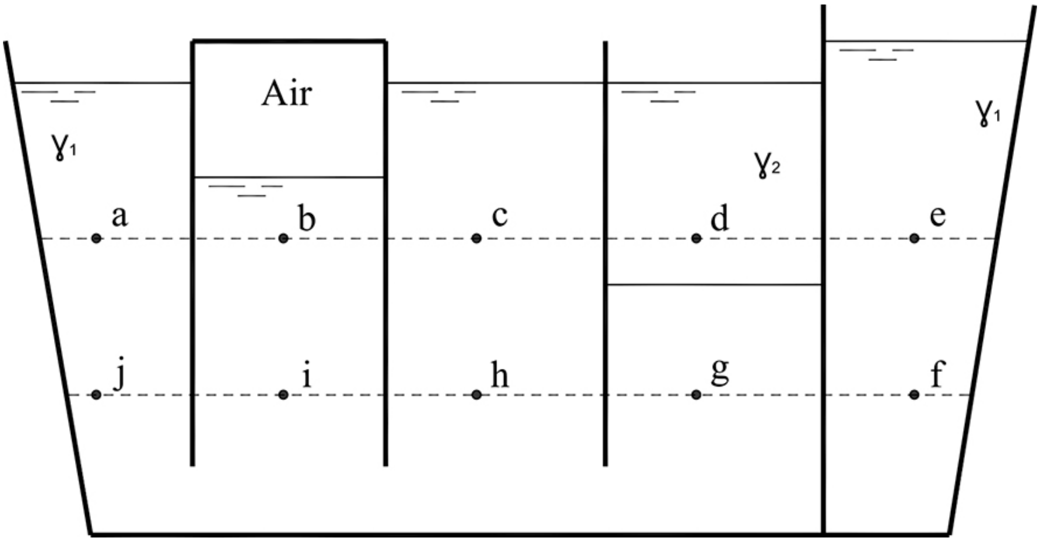


Fig. 2.3. Pressure along the same horizontal line.

THEME 3. FORCES ON A SUBMERGED SURFACE

Magnitude of the resultant force on a submerged surface

For fluids at rest the force must be perpendicular to the surface since there are no shearing stresses present. We also know that the pressure will vary linearly with depth if the fluid is incompressible.

Let the plane in which the surface lies intersect the free surface at *O* and make an angle Θ with this surface as in Fig. 3.1 (edge view and head-on view). The $x-y$ coordinate system is defined so that 0 is the origin. y axis is directed along the surface and x axes is directed into the paper. The area can have an arbitrary shape as shown. We wish to determine the direction, location, and magnitude of the resultant force acting on one side of this area due to the liquid in contact with the area.

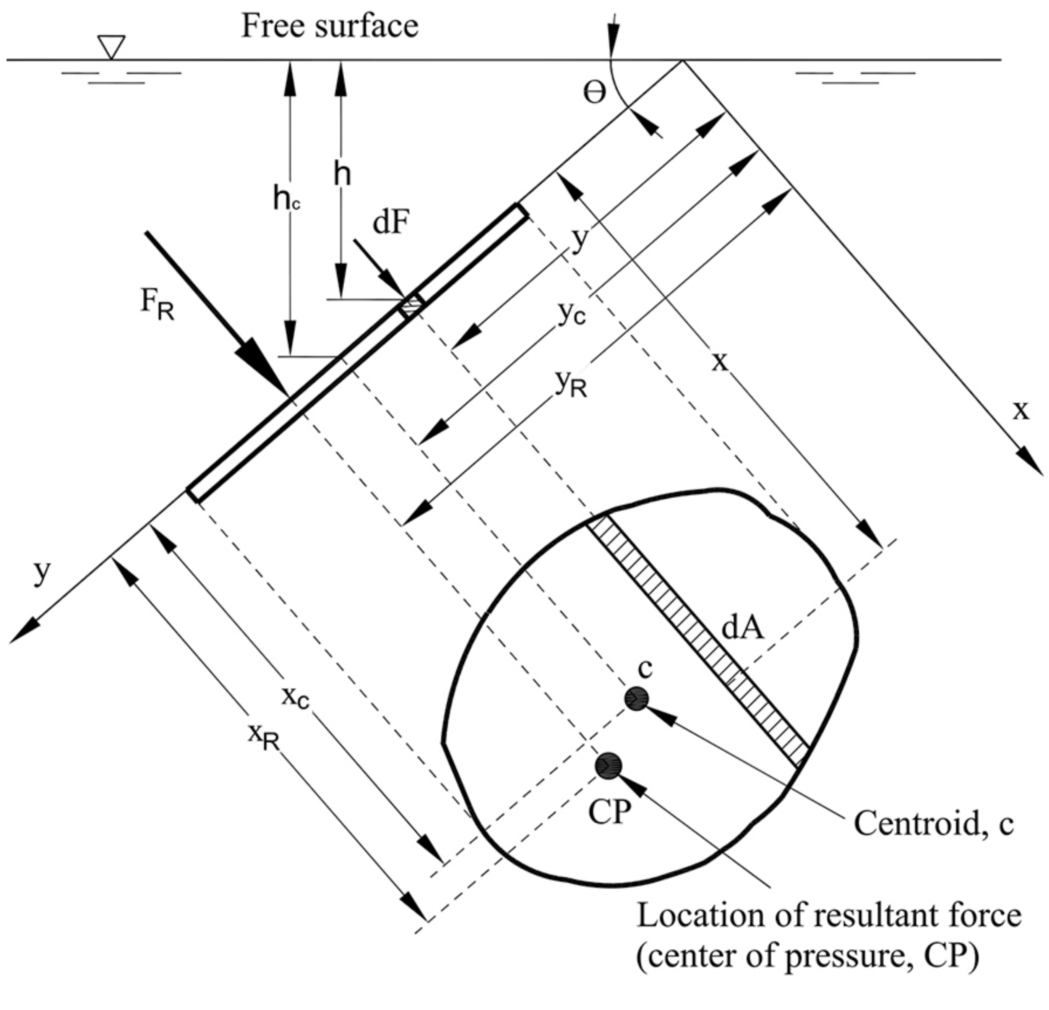


Fig. 3.1. Notation for hydrostatic force on an inclined plane surface of arbitrary shape

At any given depth, h, the pressure on the shaded strip is

$$P = \gamma \cdot h = \gamma \cdot y \cdot sin\Theta$$

where $h = y \cdot sin\Theta$. The pressure is everywhere the same along the strip because it is at the same depth down from the free surface.

So, the force acting on the strip is

$$dF = P \cdot dA = \gamma \cdot y \cdot sin\Theta \cdot dA$$

The magnitude of the resultant force can be found by integrating these differential forces over the entire surface.

$$F_R = \int_A \gamma \cdot y \cdot sin\Theta \cdot dA$$

For constant γ and Θ

$$F_R = \gamma \cdot sin\Theta \int_A y \cdot dA \qquad (3.1)$$

The integral appearing in Equation (3.1) is the first moment of the area with respect to the x axis.

Location of the centroid of an area in y direction is

$$y_c = \frac{1}{A}\int_A y \cdot dA$$

or

$$\int_A y \cdot dA = y_c \cdot A$$

Equation (3.1) can thus be written as

$$F_R = \gamma \cdot y_c \cdot A \cdot sin\Theta \qquad (3.2)$$

or more simply as

$$F_R = \gamma \cdot h_c \cdot A \qquad (3.3)$$

where h_c is the vertical distance from the fluid surface to the centroid of the area.

The magnitude of the force is independent of the angle Θ. It depends only on the specific weight of the fluid, the total area, and the depth of the centroid of the area below the surface. Equation (3.3) indicates that the magnitude of the resultant force is equal to the pressure at the centroid of the area multiplied by the total area. Since all the differential forces that were summed to obtain F_R are perpendicular to the surface, the resultant F_R must also be perpendicular to the surface.

Coordinates of the resultant force on a submerged surface

The pressure is zero gauge at the point O (at the surface). It increases linearly as we go down along the surface creating distributed pressure force which acts normal to the surface as shown in Fig. 3.2. So, the resultant force also acts normal to the surface.

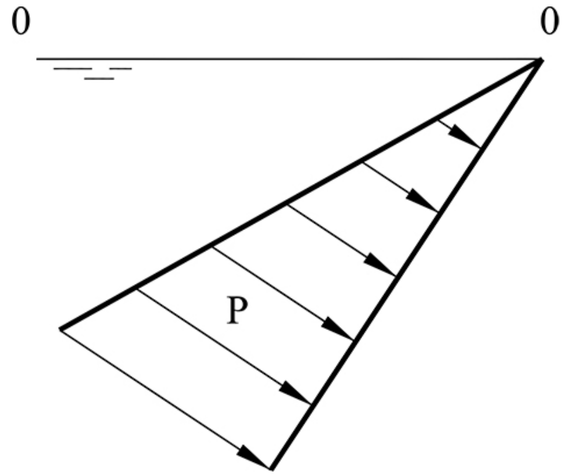


Fig. 3.2. Distributed pressure force on a submerged surface

The y coordinate, of the resultant force can be determined by summation of moments around the x axis. It means that the moment of the resultant force must equal the moment of the distributed pressure force.

$$y_R \cdot F_R = \int y \cdot dF$$

If replace the force with pressure

$$y_R \cdot F_R = \int y \cdot P \cdot dA = \int y \cdot (\gamma \cdot y \cdot sin\Theta) \cdot dA$$

or

$$y_R \cdot F_R = \gamma \cdot sin\Theta \int_A y^2 dA$$

The integral here is the second moment of the area (moment of inertia), I_x, with respect to an axis formed by the intersection of the plane containing the surface and the free surface (x axis).

According to the transfer axis theorem (the parallel axis theorem), the moment of inertia about any x-axis equals the moment of inertia about the x-axis of the centroid plus squared distance between the axis times area of the surface

$$I_x = I_{xc} + A \cdot y_c^2$$

where I_{xc} is the second moment of the area with respect to an axis passing through its *centroid* and parallel to the *x* axis.

So

$$y_R \cdot F_R = \gamma \cdot sin\Theta(I_{xc} + A \cdot y_c^2)$$

Thus, using Equation (3.2) we can write

$$y_R \cdot \gamma \cdot y_c \cdot A \cdot sin\Theta = \gamma \cdot sin\Theta(I_{xc} + A \cdot y_c^2)$$

or

$$y_R = y_c + \frac{I_{xc}}{y_c \cdot A} \quad (3.4)$$

As shown by the Figure 3.3, the resultant force does not pass through the centroid but for nonhorizontal surfaces is always below it, since, I_{xc}/y_cA, is always positive.

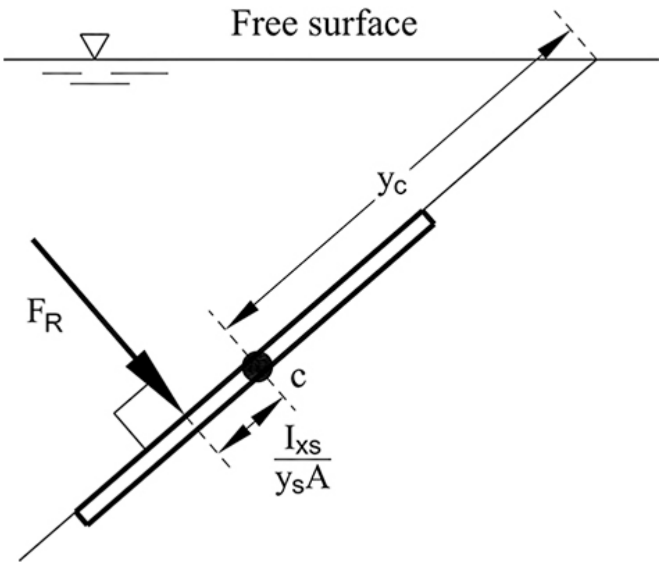


Fig. 3.3. Resultant force on a submerged surface

The *x* coordinate x_R, for the resultant force can be determined in a similar manner by summing moments about the *y* axis.

$$x_R \cdot F_R = \int x \cdot P \cdot dA = \int x \cdot (\gamma \cdot y \cdot sin\Theta) \cdot dA$$

or

$$x_R \cdot F_R = \gamma \cdot sin\Theta \int_A x \cdot y \cdot dA$$

The integral here is the product of inertia with respect to the *x* and *y* axes. Again, using the transfer axis theorem, we can write

$$x_R = x_c + \frac{I_{xyc}}{y_c \cdot A} \tag{3.5}$$

where I_{xyc} is the product of inertia with respect to an orthogonal coordinate system passing through the centroid of the area and formed by a translation of the *x–y* coordinate system. If the submerged area is symmetrical with respect to an axis passing through the centroid and parallel to either the *x* or *y* axis, the resultant force must lie along the line $x = x_c$ since I_{xyc} is identically zero in this case.

The point through which the resultant force acts is called the ***center of pressure*** (***CP***).

Centroidal coordinates and moments of inertia for some common areas are given in Fig. 3.4.

Pressure prism

Consider the pressure distribution along a vertical wall of a tank. We can represent the variation of pressure as is shown in Fig. 3.5a. The average pressure occurs at the depth *h/2* and, therefore, the resultant force acting on the rectangular area A = b · h is

$$F_R = p_{av} A = \gamma \left(\frac{h}{2}\right) A$$

which is the same result as obtained from Equation (3.3).

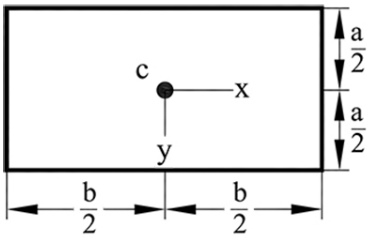
We can draw the three-dimensional representation of the pressure distribution as shown in Fig. 3.5b. This volume is called the ***pressure prism***. The resultant force acting on the rectangular surface is equal to the volume of the pressure prism

$$F_R = \frac{1}{2}(\gamma \cdot h)(b \cdot h) = \gamma \left(\frac{h}{2}\right) A \tag{3.6}$$

where $b \cdot h$ is the area of the rectangular surface, A.

The resultant force must pass through the *centroid* of the pressure prism. For the volume under consideration the centroid is located along the vertical axis of symmetry of the surface and at a distance of *h/3* above the base (since the centroid of a triangle is located at *h/3* above its base).

This result is consistent with that obtained from Equations (3.4) and (3.5).



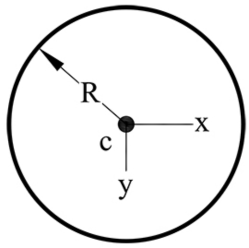
a) Rectangle

$$A = b \cdot a$$

$$I_{xc} = \frac{1}{12} b \cdot a^3$$

$$I_{yc} = \frac{1}{12} a \cdot b^3$$

$$I_{xyc} = 0$$



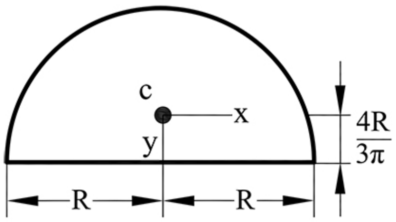
b) Circle

$$A = \pi \cdot R^2$$

$$I_{xc} = \frac{\pi \cdot R^4}{4}$$

$$I_{yc} = \frac{\pi \cdot R^4}{4}$$

$$I_{yc} = 0$$



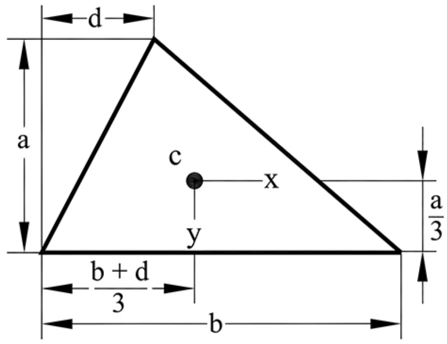
c) Semicircle

$$A = \frac{\pi \cdot R^2}{2}$$

$$I_{xc} = 0.1098 \cdot R^4$$

$$I_{yc} = 0.3927 \cdot R^4$$

$$I_{xyc} = 0$$

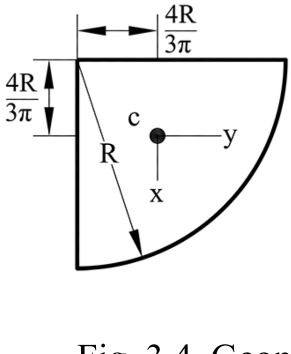


d) Triangle

$$A = \frac{a \cdot b}{2}$$

$$I_{xc} = \frac{b \cdot a^3}{36}$$

$$I_{yc} \frac{b \cdot a^2}{72} (b - 2d)$$



e) Quarter circle

$$A = \frac{\pi \cdot R^2}{4}$$

$$I_{xc} = I_{yc} = 0.05488 \cdot R^4$$

$$I_{xyc} = -0.01647 \cdot R^4$$

Fig. 3.4. Geometric properties of some common shapes

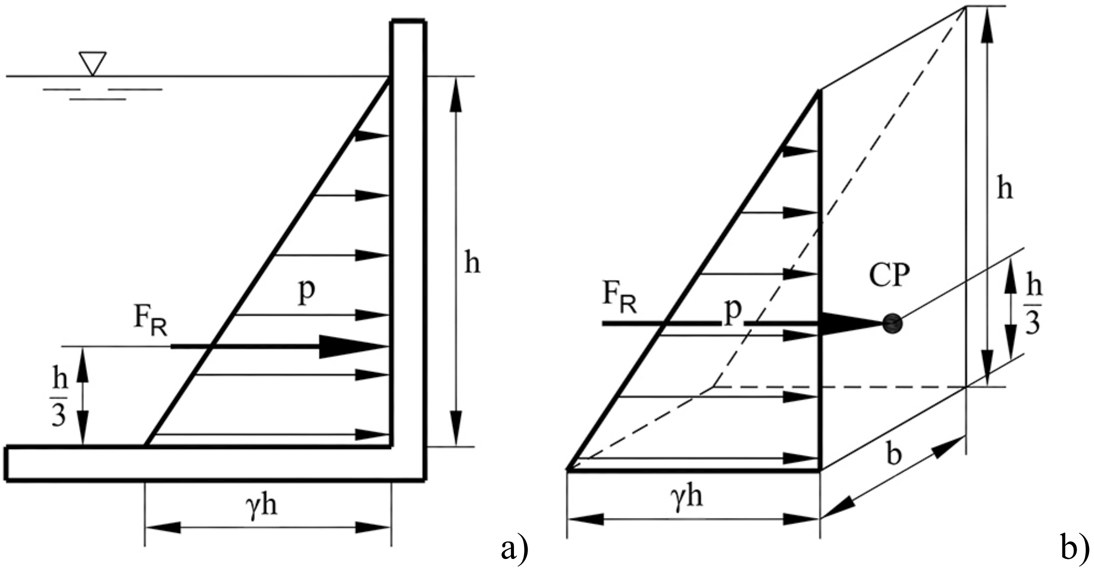


Fig. 3.5. Pressure prism for vertical rectangular area

This same graphical approach can be used for plane rectangular surfaces that do not extend up to the fluid surface, as illustrated in Fig. 3.6a. The cross section of the pressure prism is trapezoidal. The resultant force is still equal in magnitude to the volume of the pressure prism, and it passes through the centroid of the volume. Specific values can be obtained by decomposing the pressure prism into two parts, ABDE and BCD

$$F_R = F_1 + F_2.$$

The location of can be determined by summing moments about some convenient axis, such as one passing through A

$$F_R y_R = F_1 y_1 + F_2 y_2.$$

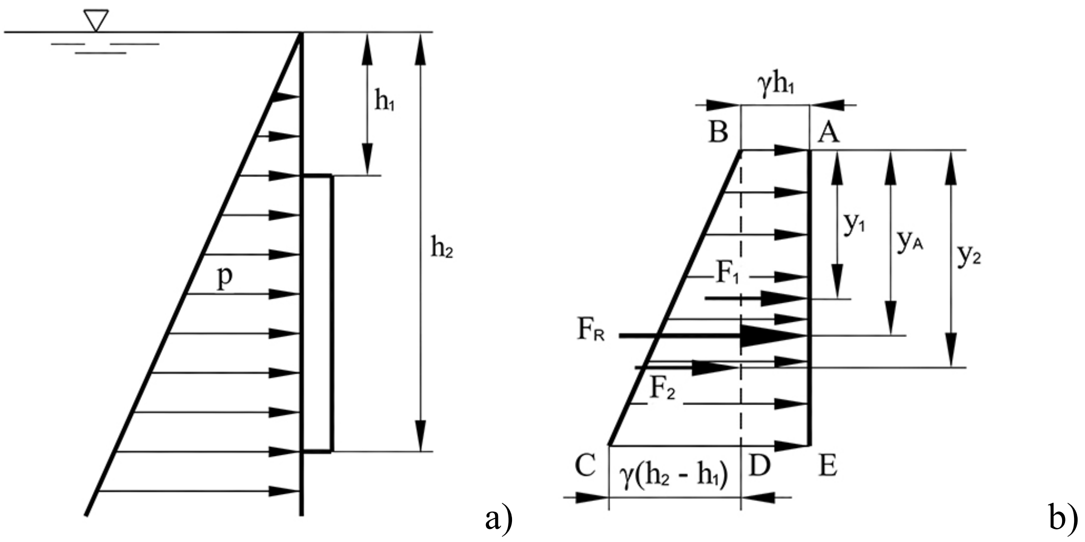


Fig. 3.6. Graphical representation of hydrostatic forces on a vertical rectangular surface

Hydrostatic force on a curved surface

Consider the curved surface shown in Fig. 3.7a. To find the resultant fluid force, F_R, acting on section *AB* it's useful to decompose it into horizontal and vertical components. To find the horizontal component of the resultant force, F_h, we can project the curved surface onto a vertical plane as shown in Fig. 3.7b and then use rules previously discussed for vertical surfaces (Equation (3.6)).

To find the vertical component of the resultant force, F_v, we can use the next equation

$$F_v = \gamma \cdot \forall.$$

where, $\forall$, the volume of fluid above the curved surface extended to the free surface (real or imaginary) as shown in Fig. 3.7c. This force acts through the centroid of $\forall$.

The magnitude of the resultant force, F_R, can be obtained from the equation

$$F_R = \sqrt{F_h + F_v}.$$

The resultant force passes through the point *O*, which can be located by summing moments about an appropriate axis.

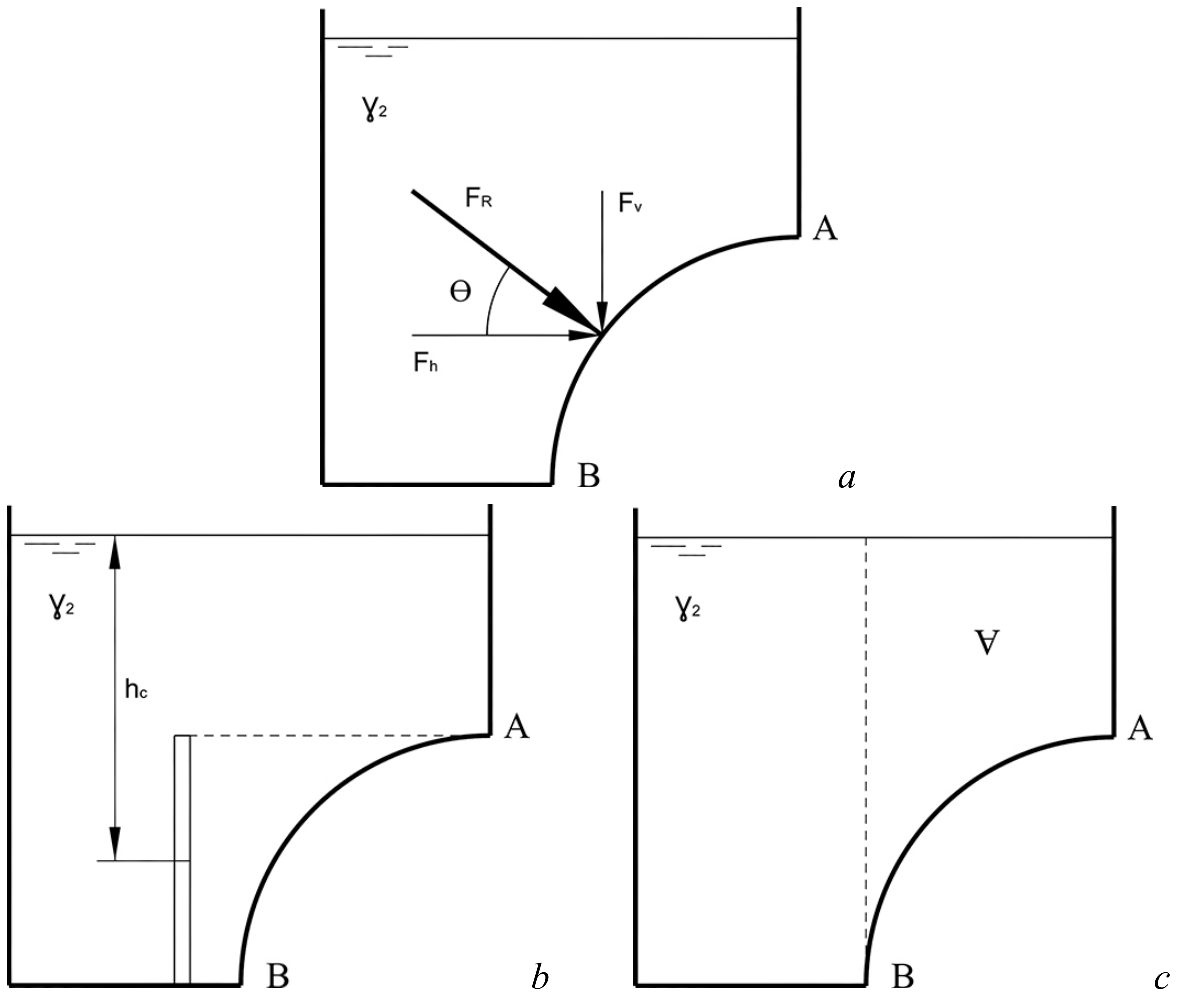


Fig. 3.7. Hydrostatic force on a curved surface

Buoyancy

When a stationary body is completely submerged in a fluid the resultant fluid force acting on the body is called the ***buoyant force***.

A net upward vertical force results because pressure increases with depth and the pressure forces acting from below are larger than the pressure forces acting from above.

Consider a body of arbitrary shape, having a volume that is immersed in a fluid as illustrated in Fig. 3.8. Take an elemental vertical slice $d\forall$ and consider the pressure forces acting on the top and on the bottom of the slice. Here *dA* is the area of the top and the bottom of the slice. The pressure goes up if we go down in the fluid

$$P = P_0 + \rho \cdot g \cdot h.$$

where, P_0, is the surface pressure at the pop of fluid.

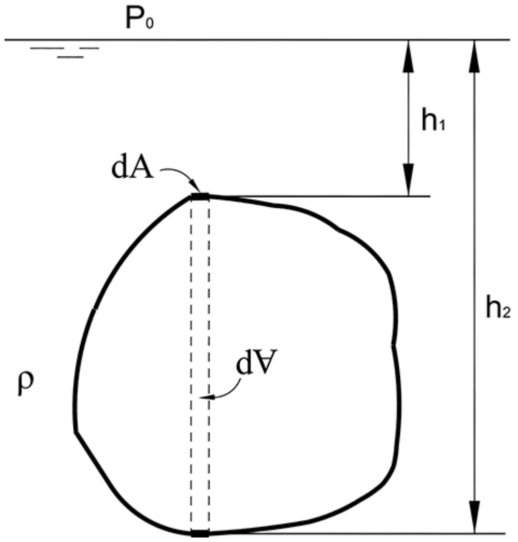


Fig. 3.8. Buoyant force on submerged bodies

Pressure is a compressive force and it always points on the surface (into the surface). The pressure on the top is pointing down and the pressure on the bottom is pointing up. The sum of the forces in the *z* direction due to pressure is

$$dF_z = (P_0 + \rho \cdot g \cdot h_2)dA - (P_0 + \rho \cdot g \cdot h_1)dA.$$

or

$$dF_z = \rho \cdot g(h_2 - h_1)dA.$$

But notice that $(h_2 - h_1)dA$ is the differential volume $d\forall$. So

$$dF_z = \rho \cdot g \cdot d\forall.$$

Now we can integrate both sides of the equation

$$\int_A dF_z = \int_A \rho \cdot g \cdot d\forall = \rho \cdot g \int_A d\forall = \rho \cdot g \cdot \forall.$$

Here ρ and g are constant, so

$$\int_A dF_z = \rho \cdot g \int_A d\forall.$$

Finally

$$F_{buoyant} = \rho \cdot g \cdot \forall. \qquad (3.7)$$

The buoyant force acts through the centroid of the displaced volume of fluid.

THEME 4. BERNOULLI'S EQUATION

Consider the motion of ***inviscid fluids***. That is, the fluid is assumed to have zero viscosity. In practice there are no inviscid fluids, since every fluid supports shear stresses when it is subjected to a rate of strain displacement. But for many flow situations the viscous effects are relatively small compared with other effects. As a first approximation it is often possible to ignore viscous effects.

The motion of each fluid particle is described in terms of its velocity vector, V, which is defined as the time rate of change of the position of the particle. The particle's velocity is a vector quantity with a magnitude (the speed, $V = |\mathbf{V}|$) and direction. As the particle moves about, it follows a particular path, the shape of which is governed by the velocity of the particle. The location of the particle along the path is a function of where the particle started at the initial time and its velocity along the path. If it is ***steady flow*** (i.e., nothing changes with time at a given location in the flow field), each successive particle that passes through a given point will follow the same path. For such cases the path is a fixed line in the x–z plane.

For steady flows each particle slides along its path, and its velocity vector is everywhere tangent to the path. The lines that are tangent to the velocity vectors throughout the flow field are called ***streamlines***.

As a fluid particle moves from one location to another, it usually experiences an acceleration or deceleration. In curvilinear motion, the velocity vectors at different points of the streamlines usually have different directions. We attach a coordinate system to the streamline and define S-direction as along streamline and N-direction as normal to the streamline (Fig 4.1). So, we have two components of acceleration:

- along a streamline (streamwise acceleration)

$$a_s = \frac{dV}{dt} = \frac{dV}{ds} \cdot \frac{dS}{dt} = \frac{dV}{ds} \cdot V,$$

where, V, is the speed of the particle, S, is the distance traveled by the particle, t, is the travel time,

- normal to a streamline (normal acceleration)

$$a_n = \frac{V^2}{R}$$

where, R, is the radius of curvature of the streamline.

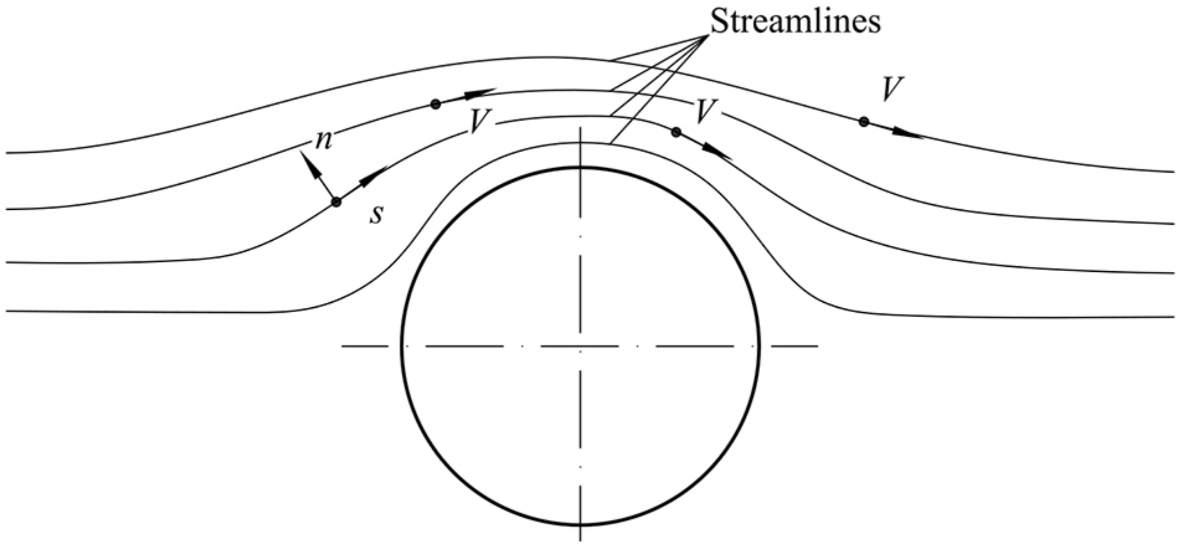


Fig. 4.1. Flow around a cylindrical object in terms of streamlines

To determine the forces necessary to produce a given flow (or conversely, what flow results from a given set of forces), we consider the free-body diagram of a small fluid particle as is shown in Fig. 4.2. The particle of interest is removed from its surroundings, and the reactions of the surroundings on the particle are indicated by the appropriate forces present. For the present case, the important forces are assumed to be gravity and pressure. Other forces, such as viscous forces and surface tension effects, are assumed negligible.

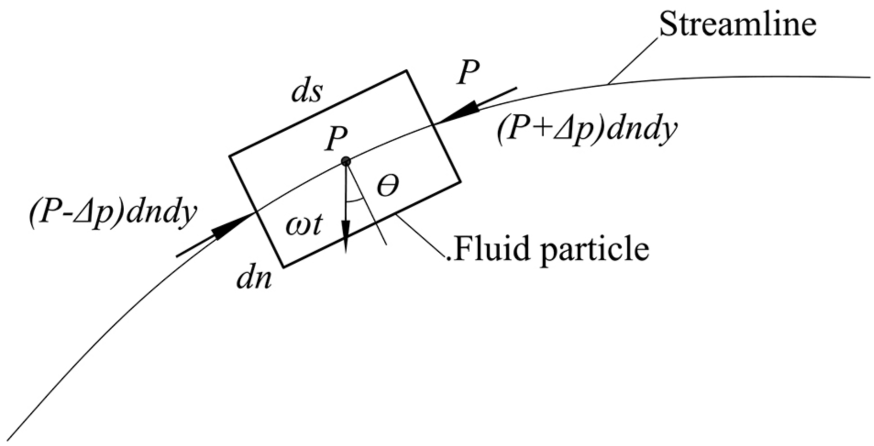


Fig. 4.2. Isolation of a small fluid particle in a flow field

The particle has dimensions ds, dn, and dy into the paper. The pressure forces on the front side of the particle, $(P + \Delta p)\cdot dn\cdot dy$, and on the back side of the particle, $(P - \Delta p)\cdot dn\cdot dy$, act along the streamline. The acceleration of gravity, g, is assumed to be constant and acts vertically, in the negative z direction, at an angle Θ relative to the normal to the streamline. So, the weight force, wt, also acts vertically, in the negative z direction.

The resultant force acting on the particle in the direction of the streamline

$$\sum \Delta F_s = \Delta m \cdot a_s = \Delta m \cdot V\frac{dV}{ds} = \rho \cdot \Delta \forall \cdot V\frac{dV}{ds} \tag{4.1}$$

where, Δm, the mass of the particle, $\Delta \forall$, the volume of the particle.

The resultant force in the s direction will consist of the projection of the weight force of the particle on the s direction and the net pressure force in the same s direction.

For the net pressure force in the s direction, we can write

$$\Delta F_{P_s} = (P_s - \Delta p_s)\cdot \Delta n \cdot \Delta y - (P_s + \Delta p_s)\cdot \Delta n \cdot \Delta y$$

or

$$\Delta F_{P_s} = \Delta n \cdot \Delta y \cdot (P_s - \Delta p_s - P_s - \Delta p_s)$$

After reductions we get

$$\Delta F_{P_s} = -2 \cdot \Delta p_s \cdot \Delta n \cdot \Delta y$$

Notice that we can write

$$\Delta p_s = \frac{dP}{ds}\cdot\frac{\Delta s}{2}.$$

So

$$\Delta F_{P_s} = -\frac{dP}{ds}\cdot \Delta s \cdot \Delta n \cdot \Delta y = -\frac{dP}{ds}\cdot \Delta \forall.$$

But, from the geometry

$$\Delta s \cdot \Delta n \cdot \Delta y = \Delta \forall.$$

Finally

$$\Delta F_{P_s} = -\frac{dP}{ds} \cdot \Delta \forall. \tag{4.2}$$

For the weight force

$$\Delta w_s = -\Delta w \cdot sin\theta = -\gamma \cdot \Delta \forall \cdot sin\theta, \tag{4.3}$$

where, *Δw*, weight force.

Taking into account the Egns. 4.2 and 4.3, we can write the resultant force acting on the particle in the *s* direction

$$\sum \Delta F_s = \Delta w_s + \Delta F_{P_s} = -\left(-\gamma \cdot sin\theta - \frac{dP}{ds}\right) \cdot \Delta \forall. \tag{4.4}$$

Now we equate the right sides of the Equations (4.1) and (4.4) to get

$$\left(-\gamma \cdot sin\theta - \frac{dP}{ds}\right) \cdot \Delta \forall = \rho \cdot \Delta \forall \cdot V \cdot \frac{dV}{ds}$$

Taking into account that

$$sin\theta = \frac{dz}{ds},$$

and also

$$V \cdot \frac{dV}{ds} = \frac{1}{2} \cdot \frac{dV^2}{ds}$$

we can write

$$-\gamma \cdot \frac{dz}{ds} - \frac{dP}{ds} = \frac{1}{2} \cdot \rho \cdot \frac{dV^2}{ds}$$

Multiply the right and left sides of the equation (multiply through) by *ds*

$$-\gamma \cdot dz - dP = \frac{1}{2} \cdot \rho \cdot dV^2.$$

Now transfer all terms of the equation to the left side and multiply the sides by -1 to get

$$dP + \frac{1}{2} \cdot \rho \cdot dV^2 + \gamma \cdot dz = 0. \tag{4.5}$$

This equation only valid though along a streamline.
Divide the equation (divide through) by ρ to get

$$\frac{dP}{\rho} + \frac{1}{2} \cdot dV^2 + g \cdot dz = 0. \qquad (4.6)$$

The next step is to integrate the Equation (4.6)

$$\int \frac{dP}{\rho} + \frac{1}{2} \cdot V^2 + g \cdot z = C. \qquad (4.7)$$

where, C, a constant that has the same value along the same streamline

Provided that the fluid is incompressible, ρ = const, we pull the constant outside the integral

$$\frac{1}{\rho} \int dP + \frac{1}{2} \cdot V^2 + g \cdot z = C. \qquad (4.8)$$

We obtain the Bernoulli's Equation with term's dimension m^2/s^2 or ft^2/s^2

$$\frac{P}{\rho} + \frac{V^2}{2} + g \cdot z = C. \qquad (4.9)$$

Bernoulli's Equation only valid with assumptions:
- the fluid is incompressible;
- the fluid is inviscid;
- the same streamline is considered (along a streamline);
- the fluid is in steady state.

We recall that the derivation takes into account only the force of weight of a fluid particle and the pressure forces. At the same time, viscous forces are neglected, which are frictional forces acting on the sides of the particle, and are consequences of its interaction with other particles of other (adjacent) streamlines.

Multiply both parts of Equation (4.9) by ρ to obtain Bernoulli's Equation with term's dimension N/m^2 or lb/ft^2

$$P + \frac{\rho \cdot V^2}{2} + \gamma \cdot z = C. \qquad (4.10)$$

where, $P,$ – static pressure, $\frac{\rho \cdot V^2}{2}$, pressure, $\gamma \cdot z,$ hydrostatic pressure, $P + \frac{\rho \cdot V^2}{2}$, stagnation pressure, $P + \frac{\rho \cdot V^2}{2} + \gamma \cdot z$, total pressure.

The stagnation pressure is the pressure at the stagnation point (the point of zero speed, where the flow stops $V = 0$)

Divide the Equation (4.10) by g to obtain Bernoulli's Equation with term's dimension meters (feet) of fluid column:

$$\frac{P}{\gamma} + \frac{V^2}{2g} + z = C, \qquad (4.11)$$

where, $\frac{P}{\gamma}$, pressure head, $\frac{V^2}{2g}$, velocity head, z, elevation head.

THEME 5. FLUID KINEMATICS

Continuum hypothesis

Fluid kinematics discusses various aspects of fluid motion without being concerned with the actual forces necessary to produce the motion. That is, we will consider the kinematics of the motion – the velocity and acceleration of the fluid, and the description and visualization of its motion.

A typical portion of fluid contains so many molecules that it becomes totally unrealistic (except in special cases) for us to attempt to account for the motion of individual molecules. Rather, we employ the ***continuum hypothesis*** and consider fluids to be made up of fluid particles that interact with each other and with their surroundings. Each particle contains numerous molecules. Thus, we can describe the flow of a fluid in terms of the motion of fluid particles rather than individual molecules.

At a given instant in time, a description of any fluid property (such as density, pressure, velocity, and acceleration) may be given as a function of the fluid's location. This representation of fluid parameters as functions of the spatial coordinates is termed a ***field representation*** of the flow.

By definition, the velocity of a particle is the time rate of change of the position vector for that particle. As illustrated in Fig. 5.1, the position of particle *A* relative to the coordinate system is given by its position vector, which (if the particle is moving) is a function of time. The time derivative of this position gives the velocity of the particle

$$\frac{dr_A}{dt} = V_A.$$

Each position vector can be represented in the form of its components (projections on the x, y, z axis), as well as a function of time t.

When doing ***field representation***, they usually use the notations shown in the Fig. 5.2 (multiply the spatial coordinates by the unit vector of the field i, j, k).

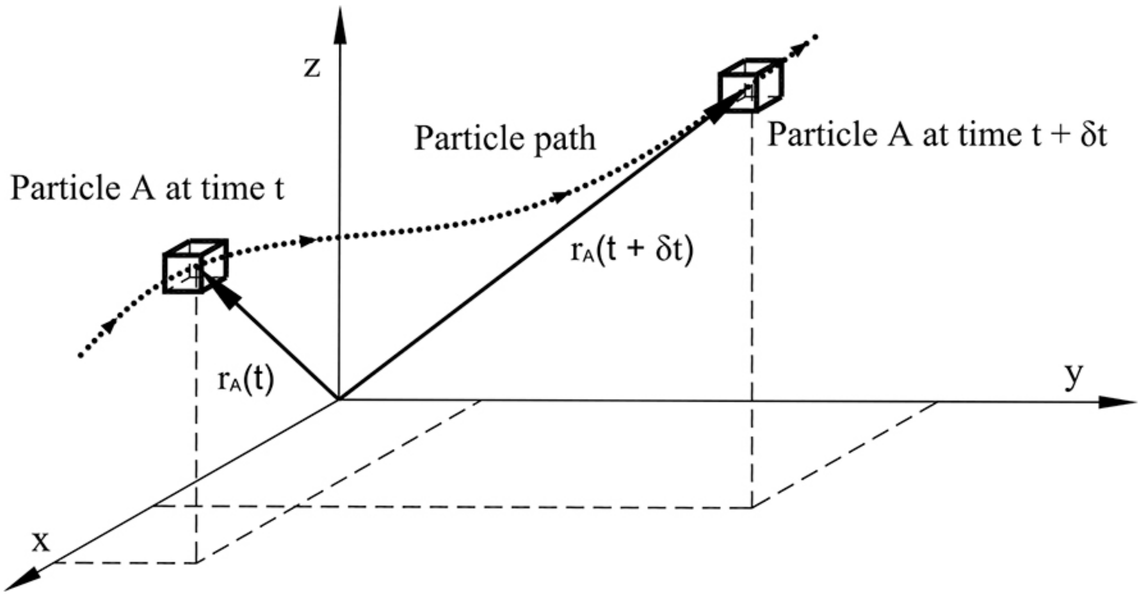


Fig. 5.1. Particle location in terms of its position vector

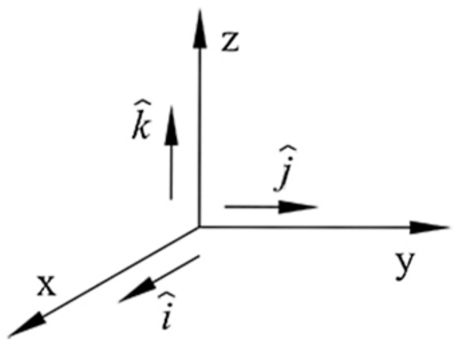


Fig. 5.2. Unit vector of the field

Thus, the velocity can be represented by a velocity vector

$$V = u(x, y, z, t) \cdot i + v(x, y, z, t) \cdot j + w(x, y, z, t) \cdot k,$$

where, *u*, *v*, *w*, projections of the velocity vector on the coordinate axis x, y, z, respectively.

By writing the velocity for all of the particles, we can obtain the field description of the velocity vector

$$\tilde{V} = \tilde{V}(x, y, z, t)$$

Since the velocity is a vector, it has both a direction and a magnitude. The magnitude of V, denoted is the speed of the fluid

$$V = |\tilde{V}| = (u^2 + v^2 + w^2)^{\frac{1}{2}}.$$

A change in velocity results in an acceleration. This acceleration may be due to a change in speed and/or direction.

Eulerian and Lagrangian approach to fluid description

There are two general approaches in analyzing fluid mechanics problems. The first method, called the ***Eulerian method***, uses the field concept introduced above. In this case, the fluid motion is given by completely prescribing the necessary properties (pressure, density, velocity, etc.) as functions of space and. From this method we obtain information about the flow in terms of *what happens at fixed points in space (Location 0 as shown in the Fig. 5.3) as the fluid flows through those points*.

The second method, called the ***Lagrangian method***, involves *following individual fluid particles (Particle A as shown in the Fig. 5.3) as they move about* and determining how the fluid properties associated with these particles change as a function of time. That is, the fluid particles are "tagged" or identified, and their properties determined as they move.

In fluid mechanics it is usually easier to use the Eulerian method to describe a flow – in either experimental or analytical investigations. There are, however, certain instances in which the Lagrangian method is more convenient.

An illustration of the difference between the Eulerian and Lagrangian descriptions can be seen in the following biological example. Each year thousands of birds migrate between their summer and winter habitats. Ornithologists study these migrations to obtain various types of important information. One set of data obtained is the rate at which birds pass a certain location on their migration route (birds per hour). This corresponds to a *Eulerian description*.

Another type of information is obtained by "tagging" certain birds with radio transmitters and following their motion along the migration route. This corresponds to a *Lagrangian description*.

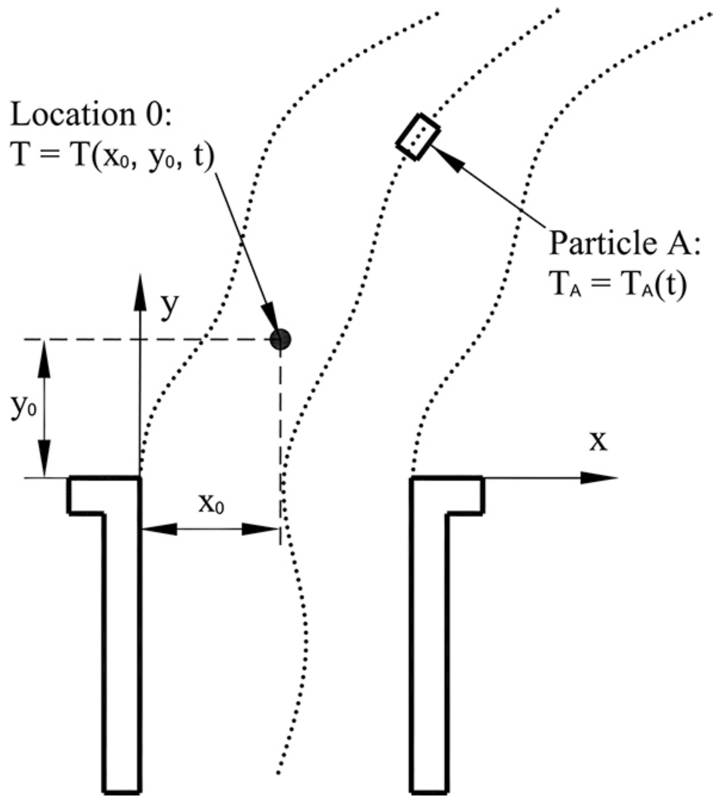


Fig. 5.3. Illustration to Eulerian and Lagrangian descriptions of temperature of flowing fluid

One-, two-, and three-dimensional flows

Generally, a fluid flow is a rather complex three-dimensional, time-dependent phenomenon $V = V(x, y, z, t) = ui + vj + wk$. In many situations, one of the velocity components may be small relative to the two other components. In situations of this kind, it may be reasonable to neglect the smaller component and assume ***two-dimensional flow***. That is, $V = ui + vj$ where u and v are functions of x and y (and possibly time, t).

It is sometimes possible to further simplify a flow analysis by assuming that two of the velocity components are negligible, leaving the velocity field to be approximated as a ***one-dimensional flow field***. That is, $V + ui$.

Steady and unsteady flows

We have assumed previously ***steady flow*** – the velocity at a given point in space does not vary with time $\partial V/\partial t = 0$. In reality, almost all flows are unsteady in some sense. That is, the velocity does vary with time. ***Unsteady***

flows are usually more difficult to analyze (and to investigate experimentally) than steady flows.

Streamlines, streaklines, and pathlines

A ***streamline*** is a line that is everywhere tangent to the velocity field. Consider x and y components u and v of velocity vector **V** as shown in Fig. 5.4. Consider also x and y components dx and dy of differential distance traveled by the particle. Big and small triangles are similar. So, we can write

$$\frac{dy}{dx} = \frac{v}{u} . \tag{5.1}$$

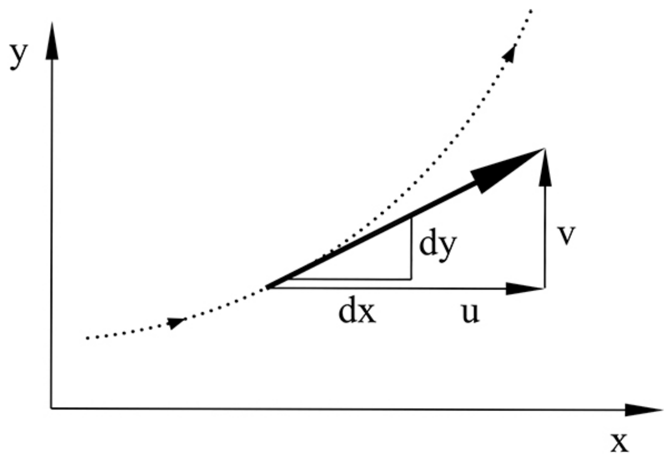


Fig. 5.4. Velocity vector for two-dimensional flow

If the velocity field is known as a function of x and y (and t if the flow is unsteady), this equation can be integrated to give the equation of the streamlines.

A ***pathline*** is the locus of points traced by a given particle as it is traced in a flow field.

A ***streakline*** is an instantaneous line whose points are occupied by all particles originating from some specified point in a flow field.

For a steady flow, all lines are the same.

Fluid acceleration field

For the Eulerian description, we describe the acceleration field as a function of position and time without actually following any particular particle.

This is analogous to describing the flow in terms of the velocity field, rather than the velocity for particular particles.

The acceleration of a particle is the time rate of change of its velocity

$$\overline{a} = \frac{d\overline{V}}{dt}, \tag{5.2}$$

where velocity vector consists of three components $\overline{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$.

Note that the velocity is a function of coordinates and time

$$\overline{V} = \overline{V}(x, y, z, t).$$

So, we expend $d\overline{V}$

$$d\overline{V} = \frac{\partial \overline{V}}{\partial x}dx + \frac{\partial \overline{V}}{\partial y}dy + \frac{\partial \overline{V}}{\partial z}dz + \frac{\partial \overline{V}}{\partial t}dt.$$

Then, according to Equation (5.2), the acceleration is

$$\overline{a} = \frac{d\overline{V}}{dt} = \frac{\partial \overline{V}}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \overline{V}}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial \overline{V}}{\partial z} \cdot \frac{dz}{dt} + \frac{\partial \overline{V}}{\partial t}. \tag{5.3}$$

Using the fact that the particle velocity components are given by

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt},$$

Equation (5.3) becomes

$$\overline{a} = u\frac{\partial \overline{V}}{\partial x} + v\frac{\partial \overline{V}}{\partial y} + w\frac{d\overline{V}}{dz} + \frac{\partial \overline{V}}{\partial t}, \tag{5.4}$$

and in component form:

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{du}{dz} + \frac{\partial u}{\partial t}, \tag{5.5a}$$

$$a_y = u\frac{\partial v}{\partial x} + \ v\frac{\partial v}{\partial y} + w\frac{dv}{dz} + \frac{\partial v}{\partial t}, \tag{5.5b}$$

$$a_z = u\frac{\partial w}{\partial x} + \ v\frac{\partial w}{\partial y} + w\frac{dw}{dz} + \frac{\partial w}{\partial t}, \tag{5.5c}$$

Equation (5.5) consists of two parts

- convective acceleration

$$\overline{a_c} = u\frac{\partial \overline{V}}{\partial x} + \ v\frac{\partial \overline{V}}{\partial y} + w\frac{dz}{dt}, \tag{5.6}$$

- local acceleration

$$\overline{a_l} = \frac{\partial \overline{V}}{\partial t}. \tag{5.7}$$

THEME 6. THE REYNOLDS TRANSPORT THEOREM

As with any matter, a fluid's behavior is governed by fundamental physical laws that are approximated by an appropriate set of equations. There are various ways that these governing laws can be applied to a fluid, including the ***system approach*** and the ***control volume approach***.

A ***system*** is a collection of matter of fixed identity (always the same atoms or fluid particles), which may move, flow, and interact with its surroundings.

A ***control volume (CV)*** is a volume in space (a geometric entity, independent of mass) through which fluid may flow.

A ***control surface (CS)*** is a surface that surrounds a control volume (it is the boundary of the control volume).

Consider a pipe with flow coming in and flow going out (Fig. 6.1). The pipe is cut at an angle Θ to the plane normal to its axis. So, A, is cross section area normal to pipe's axis and, A_Θ, is cross section area inclined at angle Θ.

The flow rate is

$$Q = V \cdot A = V \cdot A_\Theta \cdot cos\Theta.$$

We can rewrite this as a dot product

$$Q = |\overline{V}| \cdot |\overline{A}| \cdot cos\Theta = \overline{V} \cdot \overline{A}, \qquad (6.1)$$

where, $\overline{A}$, is an area vector, which always points out from the control volume (by convention).

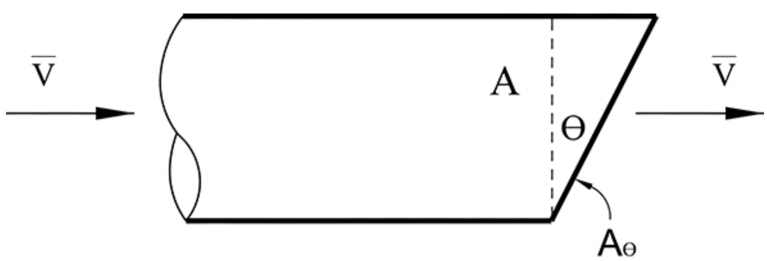


Fig. 6.1. A pipe with area vectors

Consider a flow passage with boundaries on both sides as shown in Fig. 6.2a. Dash line identifies the control volume. The control surface is its boundary. Consider the mass crossing the control surface. Flow enters at position 1 and comes out at position 2.

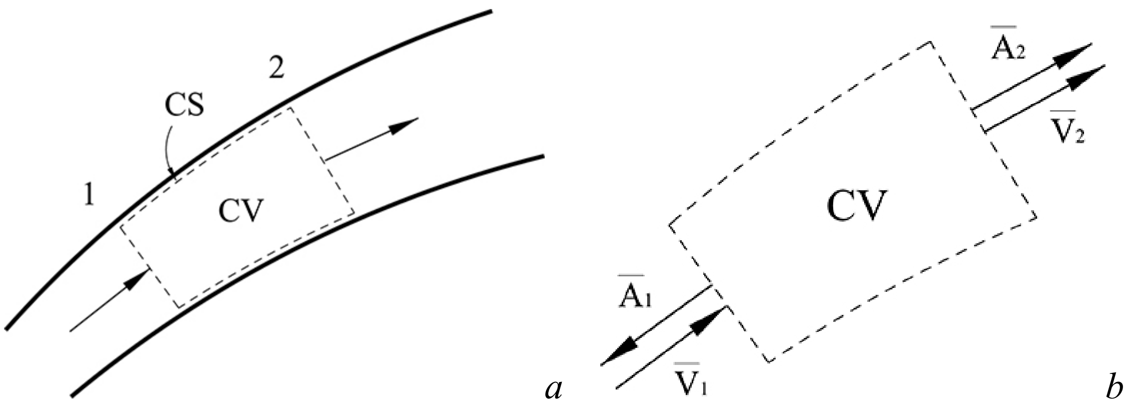


Fig. 6.2. Mass passing a control volume

If we look at the control volume separately (Fig. 6.2b) we can see velocity of mass coming in, $\overline{V_1}$, and velocity of mass going out, $\overline{V_2}$.

We can determine flowrate Q according to Equation (6.1). So, mass rate of the flow can be determined as

$$\dot{m} = \rho \cdot Q = \rho\left(\overline{V} \cdot \overline{A}\right). \qquad (6.2)$$

Vectors $\overline{V_2}$ and $\overline{A_2}$ points in the same direction. The angle between them is zero degrees (they are colinear). Using the fact that $cos0^0 = +1$ we can conclude that if the flow goes out of the control volume the dot product is positive.

Vectors $\overline{V_1}$ and $\overline{A_1}$ points in the opposite directions. The angle between them is 180^0. Using the fact that $cos180^0 = -1$ we can conclude that if the flow comes in the control volume the dot product is negative.

The net flowrate over the control surface can be written as

$$Q_{\substack{net.\\ out.}} = \sum\nolimits_{CS} \overline{V} \cdot \overline{A}, \qquad (6.3)$$

and the net mass rate over the control surface can be written as

$$\dot{m}_{\substack{net.\\uot.}} = \sum_{CS} \rho \cdot \overline{V} \cdot \overline{A}. \tag{6.4}$$

The *net* means the difference between something that goes out and something that comes in (the flowrate or mass living the control volume minus the flowrate or mass entering the control volume).

All physical laws are stated in terms of various physical parameters. Velocity, acceleration, mass, temperature, and momentum are but a few of the more common parameters. Let B represent any of these (or other) fluid parameters and *b* represent the amount of that parameter per unit mass. That is,

$$B = m \cdot b, \tag{6.5}$$

where *m* is the mass of the portion of fluid of interest.

The parameter *B* is termed an ***extensive property***, and the parameter *b* is termed an ***intensive property***. The value of B is directly proportional to the amount of the mass being considered, whereas the value of *b* is independent of the amount of mass. The amount of an extensive property that a system possesses at a given instant, B_{sys}, can be determined by adding up the amount associated with each fluid particle in the system.

If an extensive property leaves the control volume and the flow is uniform (velocity profile does not vary across the flow) it can be written (using the Equation (6.2))

$$\dot{B} = \sum_{CS} b \cdot \rho \cdot \overline{V} \cdot \overline{A}, \tag{6.6}$$

where $\dot{B}$ means with respect to time (amount in time unit)

If the flow is not uniform (velocity profile vary across the flow) than we have to be more official and use the integral form because then *V* depends on A. So

$$\dot{B} = \int_{CS} b \cdot \rho \cdot \overline{V} \cdot dA. \tag{6.7}$$

The most common property, which is meant by $\dot{B}$ is energy. If so, the units of $\dot{B}$ in *SI* system is *J* and the units of *b* is *J/kg*.

Consider a system that is a portion of fluid moving inside a pipe of variable area as shown in Fig. 6.3. Assume the flow is uniform.

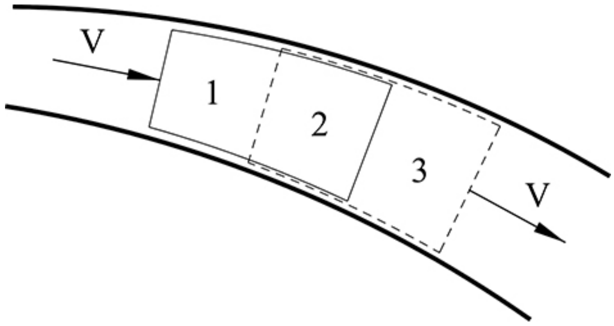


Fig. 6.3. Control volume and system for flow through a variable area pipe

At the time $t = 0$ (initial moment of time) the system is in a location represented by continuous line. In the same location is the control volume. It is represented by dot line.

At the time $t = \Delta t$ the system moves to the location represented by dash line (because the control volume is fixed in space its location, represented by dot line, does not change with time).

The next step is to write how the amount of extensive property *B* changes with respect to time in the system. It can be the next

$$\left[\frac{dB}{dt}\right]_{sys} = \lim_{\Delta t \to 0}\left[\frac{B_{t+\Delta t} - B_t}{\Delta t}\right]. \tag{6.8}$$

We can divide the area where the system moves into three regions. Region 1 – the region, which was occupied by the system at time *t* but is not occupied at time $t + \Delta t$. Region 2 – the region, which was occupied by the system at time *t* and is still occupied at time $t + \Delta t$. Region 3 – the region, which was not occupied by the system at time *t* but is already occupied at time $t + \Delta t$ (Fig. 6.3). So, at time *t* the system was located in regions 1 and 3, but at time $t + \Delta t$ the system has changed location to regions 1 and 2. Thus, we can rewrite the Equation (6.8)

$$\left[\frac{dB}{dt}\right]_{sys} = \lim_{\Delta t \to 0}\left[\frac{(B_2 + B_3)_{t+\Delta t} - (B_1 + B_2)_t}{\Delta t}\right],$$

then rearrange the terms to get

$$\left[\frac{dB}{dt}\right]_{sys} = \lim_{\Delta t \to 0}\left[\frac{B_{2(t+\Delta t)} - B_{2t}}{\Delta t}\right] = \lim_{\Delta t \to 0}\left[\frac{B_{3(t+\Delta t)} - B_{1t}}{\Delta t}\right]. \quad (6.9)$$

The first limit here is the change of amount of B with time in region 2. The region belongs to the control volume and the system is there at both times t and $t + \Delta t$. So, we can call this change the change of amount of B in the control volume

$$\lim_{\Delta t \to 0}\left[\frac{B_{2(t+\Delta t)} - B_{2t}}{\Delta t}\right] = \frac{dB_{CV}}{dt},$$

and

$$\frac{dB_{CV}}{dt} = \frac{d}{dt}\int_{CV} b \cdot \rho \cdot d\,\forall. \quad (6.10)$$

If the steady state than

$$\frac{dB_{CV}}{dt} = 0.$$

The second limit is the change of amount of B due to the system moving through the control surface. The region 3 contains the amount of B that came out of the control volume (through the control surface). The region 1 contains the amount of B that came in into the control volume (through the control surface). So, it is the net rate of change of B over the control surface.

If the flow is uniform, we can use Equation (6.6), then

$$\lim_{\Delta t \to 0}\left[\frac{B_{3(t+\Delta t)} - B_{1t}}{\Delta t}\right] = \sum_{CS} b \cdot \rho \cdot \overline{V} \cdot \overline{A}. \quad (6.11)$$

If the flow is not uniform, we can use Equation (6.7), then

$$\lim_{\Delta t \to 0}\left[\frac{B_{3(t+\Delta t)} - B_{1t}}{\Delta t}\right] = \int_{CS} b \cdot \rho \cdot \overline{V} \cdot dA. \quad (6.12)$$

But it is valid only when velocity vector $\overline{V}$ and area vector $\overline{A}$ are pointed along the same line. If not, Equation (6.1) must be taken into account. In this case, from the definition of the dot product we can write

$$V \cdot cos\Theta = \overline{V} \cdot \overline{n},$$

where $\overline{n}$ is unit vector normal to the area. So

$$\lim_{\Delta t \to 0}\left[\frac{B_{3(t+\Delta t)} - B_{1t}}{\Delta t}\right] = \int_{CS} b \cdot \rho \cdot \overline{V} \cdot \overline{n} \cdot dA. \quad (6.13)$$

Finally, using Equations (6.9 – 6.13) we can write the next

- for uniform flow

$$\left[\frac{dB}{dt}\right]_{sys} = \frac{d}{dt}\int_{CV} b \cdot \rho \cdot d\forall + \sum\nolimits_{CS} b \cdot \rho \cdot \overline{V} \cdot \overline{A}, \quad (6.14)$$

- for not uniform flow

$$\left[\frac{dB}{dt}\right]_{sys} = \frac{d}{dt}\int_{CV} b \cdot \rho \cdot d\forall + \int_{CS} b \cdot \rho \cdot \overline{V} \cdot \overline{n} \cdot dA. \quad (6.15)$$

Equations (6.14) and (6.15) are the ***Reynolds Transport Theorem.***

It can be written also in words. ***Time rate of change of B in the system equals to time rate of change of B in the control volume plus net outflow rate of B through the control surface.***

The Reynolds Transport Theorem as given in Equation (6.15) is widely used in fluid mechanics (and other areas as well). The left side of Equation (6.15) is the time rate of change of an arbitrary extensive parameter of a system. This may represent the rate of change of mass, momentum, energy, or angular momentum of the system, depending on the choice of the parameter *B*.

Because the system is moving and the control volume is stationary, the time rate of change of the amount of B within the control volume is not necessarily equal to that of the system.

The first term on the right side of Equation (6.15) represents the rate of change of B within the control volume as the fluid flows through it.

The last term in Equation (6.15) (an integral over the control surface) represents the net flowrate of the parameter B across the entire control surface. Each fluid particle or fluid mass carries a certain amount of B with it, as given by the product of B per unit mass, b, and the mass. The rate at which this B is carried across the control surface is given by the area integral term. This net rate across the entire control surface may be negative, zero, or positive depending on the particular situation involved.

THEME 7. CONSERVATION OF MASS AND CONTINUITY EQUATION

A system is defined as a collection of unchanging contents, so the ***conservation of mass*** principle for a system is simply stated as: ***Time rate of change of the system mass is equal zero***.

Let's take the Reynolds Transport Theorem and let be

$$B = \text{he mass of the system.}$$

Then, according to Equation (6.5)

$$b = \frac{B}{m} = \frac{mass}{mass} = 1.$$

In this case, the theorem says

$$\left[\frac{dB}{dt}\right]_{sys} = \left[\frac{d(mass)}{dt}\right]_{sys} = \frac{d}{dt}\int_{CV} 1 \cdot \rho \cdot d\,\forall + \int_{CS} 1 \cdot \rho \cdot \overline{V} \cdot \overline{n} \cdot dA.$$

By the definition the mass of a system is not changing, so

$$\left[\frac{dB}{dt}\right]_{sys} = \left[\frac{d(mass)}{dt}\right]_{sys} = 0.$$

Assume uniform flow for simplicity, then we obtain the ***conservation of mas equation for a control volume***

$$0 = \frac{d}{dt}\int_{CV} \rho \cdot d\,\forall + \sum_{CS} \rho \cdot \overline{V} \cdot \overline{A}. \tag{7.1}$$

If the steady state (nothing changes in control volume) and one-dimensional flow, then

$$\frac{d}{dt}\int_{CV} \rho \cdot d\,\forall = 0. \tag{7.2}$$

So, there must be

$$0 = \sum_{CS} \rho \cdot \overline{V} \cdot \overline{A}. \qquad (7.3)$$

The right side of this equation is a sum of mass flow rate in and out control volume, namely

$$\rho_{out} \cdot V_{out} \cdot A_{out} - \rho_{in} \cdot V_{in} \cdot A_{in} = m_{out} - \dot{m_{in}} = 0,$$

or

$$\dot{m_{in}} = \dot{m_{out}} \qquad (7.4).$$

It means that in a system ***the mass that comes in equals the mass that goes out***, or ***the amount of mass in a system is constant***.

Note that several assumptions were made to get Equation (7.4):
- uniform flow;
- steady state;
- one-dimensional flow.

Continuity equation

The Equation (7.1) is also called the ***continuity equation***, for a fixed, nondeforming control volume.

For steady state and one-dimensional flow Equations (7.2) and (7.3) are valid.

If the flow is incompressible

$$\rho_{in} = \rho_{out},$$

then, we can get using Equation (7.3)

$$V_{out} \cdot A_{out} = V_{in} \cdot A_{in},$$

or

$$Q_{out} = Q_{in} \qquad (7.5).$$

If the area of the flow is the same

$$A_{out} = A_{in},$$

we can write

$$V_{out} = V_{in} \qquad (7.6).$$

Note that additional assumptions were made to get Equations (7.5) and (7.6):

- flow is incompressible (for Equation (7.5));
- the area of the flow is constant (for Equation (7.6)).

Momentum equation

Newton's second law of motion for a system says: ***Time rate of change of the linear momentum of the system equals the sum of external forces acting on the system***.

Let's take the Reynolds Transport Theorem and let be

$$B = \text{momentum of the system.}$$

Since momentum is mass times velocity

$$B = \mathrm{m} \cdot \overline{V},$$

then, according to Equation (6.5)

$$b = \frac{B}{m} = \frac{\mathrm{m} \cdot \overline{V}}{m} = \overline{V}.$$

In this case, the theorem says

$$\left[\frac{dB}{dt}\right]_{sys} = \left[\frac{d(m \cdot \overline{V})}{dt}\right]_{sys} = \frac{d}{dt}\int_{CV} \rho \cdot \overline{V} \cdot d\forall + \int_{CS} \rho \cdot \overline{V} \cdot \left(\overline{V} \cdot \overline{n}\right) \cdot dA.$$

It can be written also in words. ***Time rate of change of the linear momentum of the system equals to time rate of change of the linear momentum of the contents of the control volume plus net rate of flow of linear momentum through the control surface.***

As particles of mass move into or out of a control volume through the control surface, they carry linear momentum in or out. Thus, linear momentum flow should seem no more unusual than mass flow.

According to the Newton's second law of motion for a system

$$\left[\frac{d(m \cdot \overline{V})}{dt}\right]_{sys} = \sum \overline{F},$$

where $\Sigma \overline{F}$ is summation of forces that act on the system (both body forces and surface forces).

Now, we can replace the change of the linear momentum of the system with the summation of forces that act on the system. We obtain the general form of *Momentum Equation*

$$\sum \overline{F} = \frac{d}{dt}\int_{CV} \rho \cdot \overline{V} \cdot d\forall + \int_{CS} \rho \cdot \overline{V} \cdot \left(\overline{V} \cdot \overline{n}\right) \cdot dA. \qquad (7.7)$$

If the velocity profile of the flow is uniform, then it simplifies to

$$\sum \overline{F} = \frac{d}{dt}\int_{CV} \rho \cdot \overline{V} \cdot d\forall + \sum_{CS} \rho \cdot \overline{V} \cdot \left(\overline{V} \cdot \overline{A}\right). \qquad (7.8)$$

We can also write the component form of *Momentum Equation* on the example of *x*-component

$$\sum F_x = \frac{d}{dt}\int_{CV} \rho \cdot V_x \cdot d\forall + \sum_{CS} \rho \cdot V_x \cdot \left(\overline{V} \cdot \overline{A}\right). \qquad (7.9)$$

And similar equations can be written for *y*-component and *z*-component.

We initially confine ourselves to fixed, nondeforming control volumes for simplicity. We do not consider deforming control volumes and accelerating (no inertial) control volumes. If a control volume is no inertial, the acceleration components involved (for example, translation acceleration, Coriolis acceleration, and centrifugal acceleration) require consideration.

The forces involved in the *Momentum Equation* are body and surface forces that act on what is contained in the control volume. The body force we usually always consider is the one associated with the action of gravity. We experience this body force as weight. The surface forces are basically exerted on the contents of the control volume by material just outside the control volume in contact with material just inside the control volume. For example, a wall in contact with fluid can exert a reaction surface force on the fluid it bounds. Similarly, fluid just outside the control volume can push on fluid just inside the control volume at a common interface, usually an opening in the control surface through which fluid flow occurs. An immersed object can resist fluid motion with surface forces.

Energy equation

The ***first law of thermodynamics*** is a statement of conservation of energy. The law for a system is, in words: ***Time rate of increase of the total stored energy of the system is equal to net time rate of energy addition by heat transfer into the system plus net time rate of energy addition by work transfer into the system***.

The Energy Equation written in system terms

$$\frac{D}{Dt}\int_{sys} e \cdot \rho \cdot d\,\forall = \left(\sum \dot{Q}_{in} - \sum \dot{Q}_{out}\right)_{sys} + \left(\sum \dot{W}_{in} - \sum \dot{W}_{out}\right)_{sys},$$

or

$$\frac{D}{Dt}\int_{sys} e \cdot \rho \cdot d\,\forall = \dot{Q}_{\underset{in}{net}} + \dot{W}_{\underset{in}{net}}. \tag{7.10}$$

where, e, is the total stored energy per unit mass for each particle in the system, $\dot{Q}_{\underset{in}{net}}$, the net rate of heat transfer into the system, $\dot{W}_{\underset{in}{net}}$, the net rate of work transfer into the system.

Let's take the Reynolds Transport Theorem and let B to be the total energy of the system, then

$$B = \mathrm{m} \cdot e,$$

or

$$b = \frac{B}{m} = \frac{\mathrm{E}}{m} = \mathrm{e}.$$

For the system and the contents of the coincident control volume that is fixed and nondeforming, the Reynolds Transport Theorem (Equation (6.15) with the parameter b set equal to e) allows us to conclude that

$$\frac{D}{Dt}\int_{sys} e \cdot \rho \cdot d\forall = \frac{\partial}{\partial t}\int_{CV} e \cdot \rho \cdot d\forall + \int_{CS} e \cdot \rho \cdot (\overline{V} \cdot \overline{n}) \cdot dA. \qquad (7.11)$$

In words it says: ***The time rate of increase of the total stored energy of the system is equal to the time rate of increase of the total stored energy of the contents of the control volume plus the net rate of flow of the total stored energy out of the control volume through the control surface***.

Combining Equations (7.10) and (7.11), we get the control volume formula for the first law of thermodynamics

$$\frac{\partial}{\partial t}\int_{CV} e \cdot \rho \cdot d\forall + \int_{CS} e \cdot \rho \cdot (\overline{V} \cdot \overline{n}) \cdot dA = \dot{Q}_{\underset{in}{net}} + \dot{W}_{\underset{in}{net}}. \qquad (7.12)$$

The total stored energy per unit mass, *e*, in Equation (7.12) is for fluid particles entering, leaving, and within the control volume.

Energy, *e*, is related to the internal energy per unit mass (for example heat energy), u, the potential of pressure per unit mass, P/ρ, the kinetic energy per unit mass, $V^2/2$, and the potential energy of elevation per unit mass, *gz*, by the equation

$$e = u + \frac{P}{\rho} + \frac{V^2}{2} + gz. \qquad (7.13)$$

When the equation for total stored energy (Equation (7.12)) is considered with Equation (7.13), we obtain the ***Energy equation***:

$$\frac{\partial}{\partial t}\int_{CV} e \cdot \rho \cdot d\forall + \int_{CS} \left(u + \frac{P}{\rho} + \frac{V^2}{2} + gz\right) \cdot \rho \cdot (\overline{V} \cdot \overline{n}) \cdot dA =$$
$$= \dot{Q}_{\underset{in}{net}} + \dot{W}_{\underset{in}{net}}. \qquad (7.14)$$

If assume:

- steady state flow;
- incompressible flow;
- frictionless flow (inviscid fluid),

we get

$$\frac{\partial}{\partial t}\int_{CV} e \cdot \rho \cdot d\,\forall = 0.$$

If assume additionally (for now) $\dot{W}_{\underset{in}{net}} = 0$ (no net work coming in) and knowing that

$$\dot{m} = \rho \cdot \left(\overline{V} \cdot \overline{n}\right) \cdot dA,$$

rearranging the terms in Equation (7.14) we get

$$\dot{m}\left[u_{out} - u_{in} + \frac{P_{out}}{\rho} - \frac{P_{in}}{\rho} + \frac{V_{out}^2}{2} - \frac{V_{in}^2}{2} + g(z_{out} - z_{in})\right] = \dot{Q}_{\underset{in}{net}}. \quad (7.15)$$

Dividing both sides of the Equation (7.15) by $\dot{m}$ and rearranging the terms once more we can obtain

$$\frac{P_{out}}{\rho} + \frac{V_{out}^2}{2} + gz_{out} = \frac{P_{in}}{\rho} + \frac{V_{in}^2}{2} + gz_{in} - \left(u_{out} - u_{in} - q_{\underset{in}{net}}\right), \quad (7.16)$$

where

$$q_{\underset{in}{net}} = \frac{\dot{Q}_{\underset{in}{net}}}{\dot{m}}$$

is the heat transfer rate per mass flowrate, or the heat transfer per unit of mass.

Equation (7.16) is applicable to one-dimensional flow of a single stream of fluid between two sections or flow along a streamline between two sections.

If the steady, incompressible flow we are considering also involves negligible viscous effects (frictionless flow), then the ***Bernoulli's equation*** (Equation (4.9)) can be used to describe what happens between two sections in the flow as

$$\frac{P_{out}}{\rho} + \frac{V_{out}^2}{2} + gz_{out} = \frac{P_{in}}{\rho} + \frac{V_{in}^2}{2} + gz_{in}, \quad (7.17)$$

A comparison of Equations (7.16) and (7.17) prompts us to conclude that

$$\left(u_{out} - u_{in} - q_{\substack{net \\ in}}\right) = 0$$

when the steady incompressible flow is frictionless. For steady incompressible flow with friction, we learn from experience (second law of thermodynamics) that

$$\left(u_{out} - u_{in} - q_{\substack{net \\ in}}\right) > 0$$

In Equations (7.16) and (7.17), we can consider the combination of variables

$$\frac{P}{\rho} + \frac{V^2}{2} + gz$$

as equal to useful or available energy. Thus, from inspection of Equations (7.16) and (7.17), we can conclude that $u_{out} - u_{in} - q_{net.in}$ represents the loss of useful or available energy that occurs in an incompressible fluid flow because of friction. In equation form we have

$$u_{out} - u_{in} - q_{\substack{net \\ in}} = losses.$$

For a frictionless flow, Equations (7.16) and (7.17) tell us that loss equals zero.

It is often convenient to express Equation (7.16) in terms of loss as

$$\frac{P_{out}}{\rho} + \frac{V_{out}^2}{2} + gz_{out} = \frac{P_{in}}{\rho} + \frac{V_{in}^2}{2} + gz_{in} - losses. \tag{7.18}$$

In Equation (7.12) (the control volume formula for the first law of thermodynamics) we assumed the net rate of work transfer into the system $\dot{W}_{\substack{net \\ in}} = 0$. But an important group of fluid mechanics problems involves one-dimensional, incompressible, steady-in-the-mean flow with *friction* and *shaft work*. Included in this category are constant density flows through pumps, blowers, fans, and turbines. For this kind of flow, we have to bring back the work term. Let's bring it as the net rate of *shaft work* transfer into the system and label it with $\dot{W}_{\substack{shaft\\net.in}}$.

I this case

$$w_{\substack{shaft\\net.in}} = \frac{\dot{W}_{\substack{shaft\\net.in}}}{\dot{m}}.$$

Taking this into account we can rewrite Equation (7.18)

$$\frac{P_{out}}{\rho} + \frac{V_{out}^2}{2} + gz_{out} = \frac{P_{in}}{\rho} + \frac{V_{in}^2}{2} + gz_{in} + w_{\substack{shaft\\net.in}} - losses. \quad (7.19)$$

This is a form of the energy equation for steady-in-the-mean flow that is often used for incompressible flow problems. It is sometimes called the ***Mechanical Energy equation*** or the ***Extended Bernoulli's equation***.

We can divide the Equation (7.16) by g to get

$$\frac{P_{out}}{\gamma} + \frac{V_{out}^2}{2g} + z_{out} = \frac{P_{in}}{\gamma} + \frac{V_{in}^2}{2g} + z_{in} + h_s - h_l, \quad (7.20)$$

where, h_s, is called the ***shaft work head*** and, h_l, is called the ***head loss***

$$h_s = \frac{w_{\substack{shaft\\net.in}}}{g}, \quad h_l = \frac{losses}{g}.$$

If there is a pump or a turbine in the flow the work transfer term can be (without taking into account their efficiency)

- for a pump

$$\dot{W}_P = \gamma \cdot Q \cdot H_P \; (watts),$$

- for a turbine

$$\dot{W}_T = \gamma \cdot Q \cdot H_T \; (watts).$$

When taking into account efficiency (which is always less than one) it can be determined

- for a pump

$$\eta_P = \frac{\gamma \cdot Q \cdot H_P}{\dot{W}_P},$$

- for a turbine

$$\eta_T = \frac{\dot{W}_T}{\gamma \cdot Q \cdot H_T}.$$

If a turbine is in the control volume, the work transfer term is negative because it is associated with shaft work out of the control volume. For a pump in the control volume, the work transfer term is positive because it is associated with shaft work into the control volume.

To summarize it, there are three forms of the ***Energy equation***.

The first (main) form in SI in (*N·m/s*), or (*J/s*), or (*W*)

$$\dot{m}\left[\frac{P_1}{\rho} + \frac{V_1^2}{2} + g \cdot z_1\right] + \dot{W}_P = \dot{m}\left[\frac{P_2}{\rho} + \frac{V_2^2}{2} + g \cdot z_2\right] + \dot{W}_T + losses. \quad (7.21)$$

The second (the first form divided by $\dot{m}$) form in SI in (*N·m/kg*), or (*N·m/s* divided by *kg/s*)

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + g \cdot z_1 + w_P = \frac{P_2}{\rho} + \frac{V_2^2}{2} + g \cdot z_2 + w_T + losses. \quad (7.22)$$

The third (the second form divided by *g*) form in SI in (*m*), in BG in (*ft*)

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_P = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_T + h_l, \quad (7.23)$$

THEME 8. CONSERVATION OF MASS IN DIFFERENTIAL FORM

We will take as our control volume the small, stationary cubical element of fluid shown in Fig. 8.1.

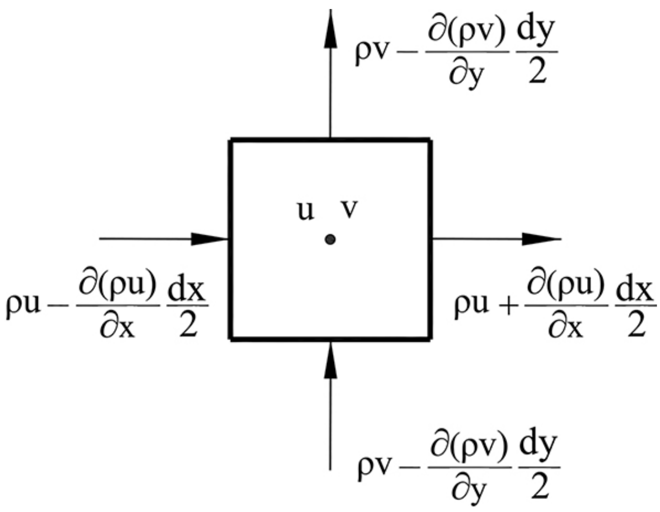


Fig. 8.1. A differential element for the development of conservation of mass equation

∂

The element has length *dx* in *x* direction and *dy* in *y* direction. Assume that the length in z direction is 1 and we have two-dimensional case (2D).

In the middle of the element velocity is *u* in *x* direction and *v* in *y* direction. We are looking at the mass that goes in and out of this small element of fluid *dx* by *dy*. The mass is coming in from the left and the mass is going out to the right. The mass also is coming in from the bottom and the mass going out through the top.

Conservation of mass in simplistic terms is

$$\frac{\partial m}{\partial t} = \dot{m}_{in} - \dot{m}_{out}. \tag{8.1}$$

In words it is: ***The time rate of change of mass in the element is equal to the amount of mass that came in minus the amount of mas that went out***.

The rate of mass flow through the surfaces of the element can be obtained by considering the flow in each of the coordinate directions separately. If we let $\rho \cdot u$ represent the *x* component of the mass rate of flow per unit area at the center of the element, then on the right face

$$\rho \cdot u|_{x+\frac{dx}{2}} = \rho \cdot u + \frac{\partial(\rho \cdot u)}{\partial x} \cdot \frac{dx}{2},$$

and on the left face

$$\rho \cdot u|_{x-\frac{dx}{2}} = \rho \cdot u - \frac{\partial(\rho \cdot u)}{\partial x} \cdot \frac{dx}{2}.$$

The same thing we can do in *y* direction. If we let $\rho \cdot v$ represent the *y* component of the mass rate of flow per unit area at the center of the element, then on the bottom face

$$\rho \cdot v|_{y-\frac{dy}{2}} = \rho \cdot v - \frac{\partial(\rho \cdot v)}{\partial y} \cdot \frac{dy}{2},$$

and on the top face

$$\rho \cdot v|_{y+\frac{dy}{2}} = \rho \cdot v + \frac{\partial(\rho \cdot v)}{\partial y} \cdot \frac{dy}{2}.$$

Using the fact that in order to obtain the mass the mass rate of flow per unit area must be multiplied by the area

$$\dot{m} = \rho \cdot V \cdot A.$$

So, the conservation of mass (Equation (8.1)) can be written as

$$\frac{\partial}{\partial t}(\rho \cdot dx \cdot dy \cdot 1) \\ = \left[\rho \cdot u - \frac{\partial(\rho \cdot u)}{\partial x} \cdot \frac{dx}{2}\right] dy \cdot 1 + \left[\rho v - \frac{\partial(\rho \cdot v)}{\partial y} \cdot \frac{dy}{2}\right] dx \cdot 1 \\ - \left[\rho \cdot u + \frac{\partial(\rho \cdot u)}{\partial x} \cdot \frac{dx}{2}\right] dy \cdot 1 - \left[\rho v + \frac{\partial(\rho \cdot v)}{\partial y} \cdot \frac{dy}{2}\right] dx \cdot 1.$$

After simplifying we can obtain

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho \cdot u) - \frac{\partial}{\partial y}(\rho \cdot v). \quad (8.2)$$

If flow is incompressible, then ρ is constant (it does not change with time) so the left side of the Equation (8.2) is zero. We can also pull ρ out of the partial sign and then divide by ρ and multiply by –1. Finally, we get two-dimensional ***Conservation of mass equation***

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{8.3}$$

To make it three-dimensional ***Conservation of mass equation*** we need to add one more term in z direction

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{8.4}$$

It means that any point in the flow field, if it is incompressible fluid, Equations (8.3) and (8.4) must be satisfied. Sum of partials of velocity with respect to their directions must be equal zero. It applies in every point in a flow field. Both equations apply to both steady and unsteady flow of incompressible fluids.

Note that Equation ***(7.2) is applicable for a control volume whereas*** Equation***s (8.3) and (8.4) are applicable for a point in space.***

Linear momentum equation in differential form

To develop the differential momentum equations, we can start with the linear momentum equation. Consider a differential system of mass δm and Newton's law to this mass system. We take a differential mass element in a flow field and watch how it moves through the flow field. The summation of differential forces acting on the element equal to the time rate of change of momentum (capital D means material derivative)

$$\delta\overline{F}=\frac{D}{Dt}\left(\overline{V}\cdot\delta m\right).$$

Because we consider a system the mass is constant (by the definition of the system). So, we can pull mass out of derivative. Time rate of change of velocity is acceleration, so

$$\delta\overline{F} = \delta m \cdot \overline{a}. \tag{8.5}$$

The forces on the left-hand side can be body forces that distributed throughout the body uniformly (typically the weight) and surface forces (can be pressure, which is normal stress, can be friction, which is shearing stresses that act on the sides of the element).

For our purpose, the only body force, of interest is the weight of the element, which can be expressed as

$$\delta\overline{F}_b = \delta m \cdot \overline{g}. \tag{8.6}$$

In component form

$$\delta F_{bx} = \delta m \cdot g_x.$$
$$\delta F_{by} = \delta m \cdot g_y.$$
$$\delta F_{bz} = \delta m \cdot g_z.$$

Surface forces act on the element as a result of its interaction with its surroundings.

At any arbitrary location within a fluid mass, the force acting on a small area, that lies in an arbitrary surface, can be represented by δF_s, as is shown in Fig. 8.2. This force can be resolved into three components, δF_n, δF_1, and δF_2, where δF_n is normal to the area, δA, and δF_1, and δF_2 are parallel to the area and orthogonal to each other. The normal stress is defined as

$$\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}.$$

and the shearing stresses are defined as

$$\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A}.$$
$$\tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}.$$

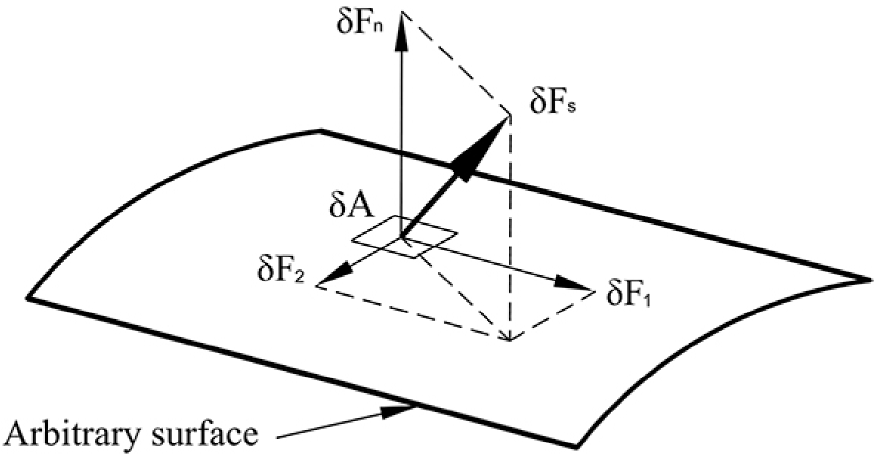


Fig. 8.2. Components of force acting on an arbitrary differential area

The intensity of the force per unit area at a point in a body can thus be characterized by a normal stress and two shearing stresses, if the orientation of the area is specified.

For the rectangular coordinate system shown in Fig. 8.3 we choose to consider the stresses acting on planes parallel to the coordinate planes. On the plane ABCD of Fig. 8.3a, which is parallel to the *y–z* plane, the normal stress is denoted σ_{xx} and the shearing stresses are denoted as τ_{xy} and τ_{xz}. To easily identify the particular stress component, we use a double subscript notation. The first subscript indicates the direction of the normal to the plane on which the stress acts, and the second subscript indicates the direction of the stress. Thus, normal stresses have repeated subscripts, whereas the subscripts for the shearing stresses are always different.

We define the positive direction for the stress as the positive coordinate direction on the surfaces for which the outward normal is in the positive coordinate direction. This is the case illustrated in Fig. 8.3a where the outward normal to the area ABCD is in the positive *x* direction. The positive directions for σ_{xx}, τ_{xy} and τ_{xz} are as shown in Fig. 8.3a.

If the outward normal points in the negative coordinate direction, as in Fig. 8.3b for the area $A'B'C'D'$, then the stresses are considered positive if directed in the negative coordinate directions. Thus, the stresses shown in Fig. 8.3b are considered to be positive when directed as shown.

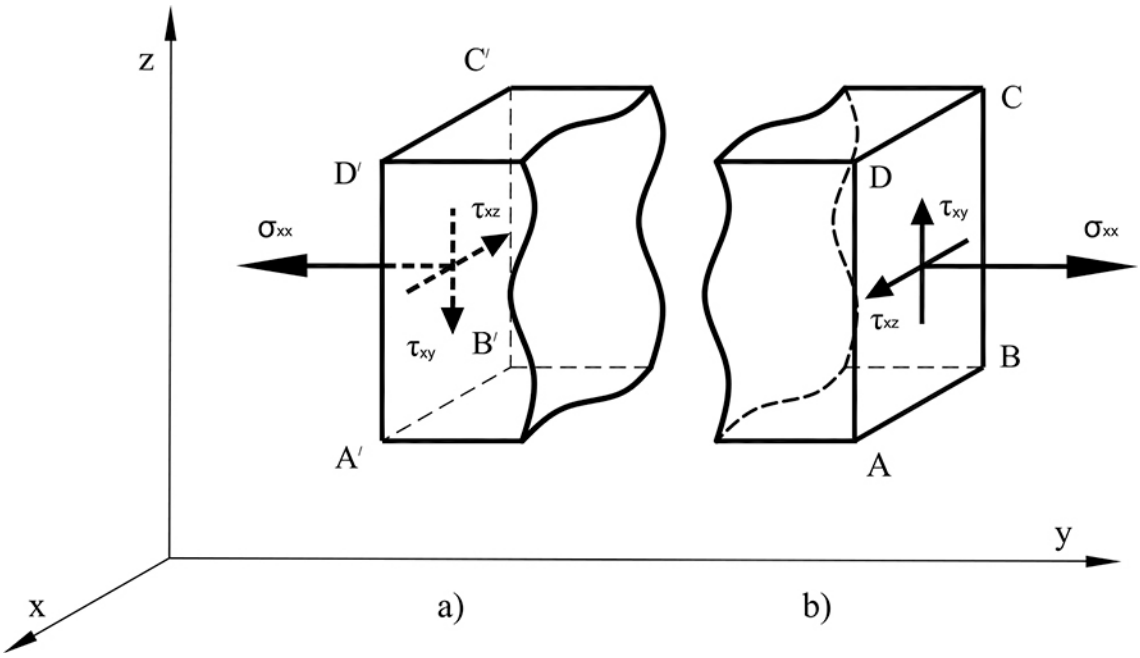


Fig. 8.3. Double subscript notation for stresses

We now can express the surface forces acting on a small cubical element of fluid in terms of the stresses acting on the faces of the element as shown in Fig. 8.4.

It is expected that in general the stresses will vary from point to point within the flow field. Thus, we can express the stresses on the various faces in terms of the corresponding stresses at the center of the element of Fig. 8.4 and their gradients in the coordinate directions. For simplicity only the forces in the *x* direction are shown.

Note that the stresses must be multiplied by the area on which they act to obtain the force.

Summing all these forces in the *x* direction yields

$$\delta F_{sx} = \left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \cdot \frac{\delta x}{2}\right] \delta y \cdot \delta z - \left[\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x} \cdot \frac{\delta x}{2}\right] \delta y \cdot \delta z$$
$$+ \left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \cdot \frac{\delta y}{2}\right] \delta x \cdot \delta z - \left[\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y} \cdot \frac{\delta y}{2}\right] \delta x \cdot \delta z$$
$$+ \left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \cdot \frac{\delta z}{2}\right] \delta x \cdot \delta y - \left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z} \cdot \frac{\delta z}{2}\right] \delta x \cdot \delta y.$$

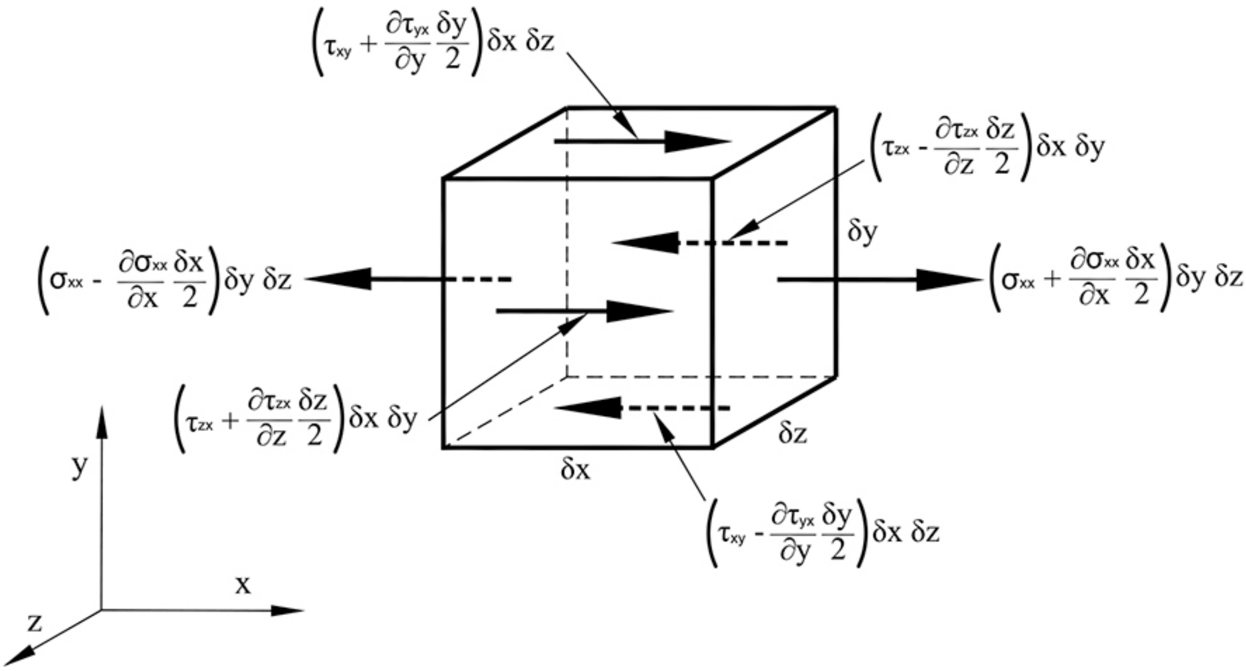


Fig. 8.4. Surface forces in the x direction acting on a fluid element

After simplifying we can get

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x \cdot \delta y \cdot \delta z \qquad (8.7a)$$

In a similar manner the resultant surface forces in the *y* and *z* directions can be obtained and expressed as

$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\delta x \cdot \delta y \cdot \delta z \qquad (8.7b)$$

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x \cdot \delta y \cdot \delta z \qquad (8.7c)$$

The expressions for the body and surface forces can now be used in conjunction with Equation (8.5) to develop the equations of motion. In component form Equation (8.5) can be written as

$$\delta F_x = \delta m \cdot a_x.$$
$$\delta F_y = \delta m \cdot a_y.$$
$$\delta F_z = \delta m \cdot a_z.$$

where $\delta m = \rho \cdot \delta x \cdot \delta y \cdot \delta z$, and the acceleration components are given by Equation (5.4). It now follows (using Equations (8.5) in component form and (8.7) also in component form for the forces on the element) that

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \qquad (8.8a)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \qquad (8.8b)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \qquad (8.8c)$$

where δx, δy, δz the element volume cancels out.

Equations (8.8) are the ***general differential equations of motion for a fluid***.

Shearing stresses develop in a moving fluid because of the viscosity of the fluid. We know that for some common fluids, such as air and water, the viscosity is small, and therefore it seems reasonable to assume that under some circumstances we may be able to simply neglect the effect of viscosity (and thus shearing stresses). Flow fields in which the shearing stresses are assumed to be negligible are said to be ***inviscid***, ***nonviscous***, or ***frictionless***. These terms are used interchangeably.

For fluids in which there are no shearing stresses the normal stress at a point is independent of direction – that is $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$. In this instance we define the pressure, p, as the negative of the normal stress so that, as indicated by the Fig. 8.4

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

The negative sign is used so that a compressive normal stress (which is what we expect in a fluid) will give a positive value for p.

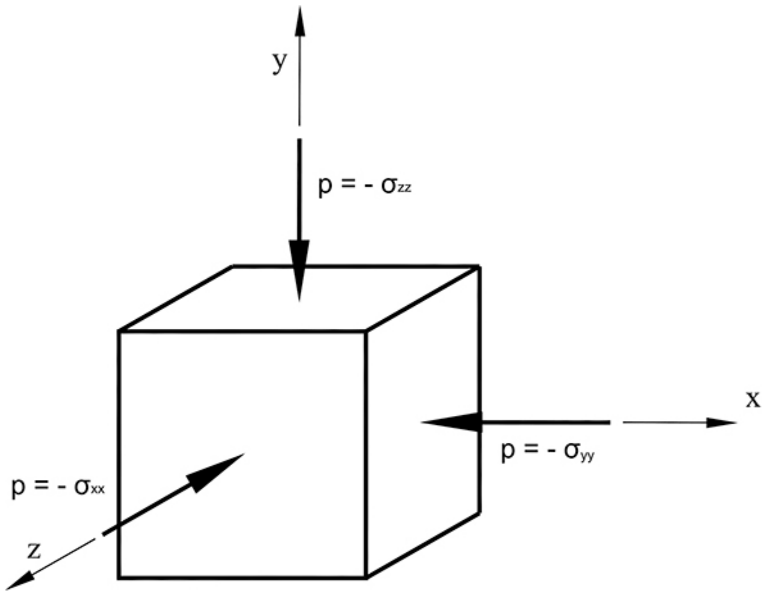


Fig. 8.5. The pressure and the normal stress

For an inviscid flow in which all the shearing stresses are zero, and the normal stresses are replaced by the general equations of motion (Equations (8.8)) reduce to

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \qquad (8.9a)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \qquad (8.9b)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \qquad (8.9c)$$

These equations are commonly referred to as ***Euler's equations of motion***. They are non-linear Partial Differential Equations (PDEs), which are difficult to solve. They are usually required to be solved for u, v, w and p. as a function of t, x, y and z.

If there are three equations (one for x, one for y and one for z) and four unknown u, v, w and p, we need additional equation to solve the system. The additional equation that can help is usually ***Continuity equation***.

Viscous flow

Now let's incorporate viscous effects into the general differential equations of motion for a fluid. For incompressible Newtonian fluids it is known that the stresses are linearly related to the rates of deformation and for normal stresses can be expressed in Cartesian coordinates as

$$\sigma_{xx} = -p + 2\mu \cdot \frac{\partial u}{\partial x}, \tag{8.10a}$$

$$\sigma_{yy} = -p + 2\mu \cdot \frac{\partial v}{\partial y}, \tag{8.10b}$$

$$\sigma_{zz} = -p + 2\mu \cdot \frac{\partial w}{\partial z}, \tag{8.10c}$$

for shearing stresses

$$\tau_{xy} = \tau_{yx} = \mu \cdot \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{8.11a}$$

$$\tau_{yz} = \tau_{zy} = \mu \cdot \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \tag{8.11b}$$

$$\tau_{zx} = \tau_{xz} = \mu \cdot \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right). \tag{8.11c}$$

where p is the pressure, the negative of the average of the three normal stresses

$$-p = \frac{1}{3} \cdot \left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right).$$

For viscous fluids in motion the normal stresses are not necessarily the same in different directions, thus, the need to define the pressure as the average of the three normal stresses. For fluids at rest, or frictionless fluids, the normal stresses are equal in all directions.

The stresses as defined by Equations (8.10) and (8.11) can be substituted into the differential equations of motion (Equations (8.8)) and simplified by using the continuity equation for incompressible flow (Equation (8.4)). The results are

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho\cdot g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \quad (8.12a)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho\cdot g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \quad (8.12b)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)= \\ =-\frac{\partial p}{\partial z}+\rho\cdot g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right). \quad (8.12c)$$

We have rearranged the equations so that the acceleration terms are on the left side and the force terms are on the right. These equations are commonly called the ***Navier-Stokes equations***.

These three equations of motion, when combined with the conservation of mass equation (Equation (8.4)), provide a complete mathematical description of the flow of incompressible Newtonian fluids. We have four equations and four unknowns *u*, *v*, *w* and *p* and therefore the problem is "well-posed" in mathematical terms. Unfortunately, because of the general complexity of the Navier–Stokes equations (they are nonlinear, second-order, partial differential equations), they are not amenable to exact mathematical solutions except in a few instances. However, in those few instances in which solutions have been obtained and compared with experimental results, the results have been in close agreement. Thus, the Navier-Stokes equations are considered to be the governing differential equations of motion for incompressible Newtonian fluids.

Note that the Euler's equations do not include viscosity, whereas the Navier-Stokes equations include viscosity.

THEME 9. VISCOUS FLOW IN PIPES. LAMINAR FLOW

Although not all conduits used to transport fluid from one location to another are round (circular) in cross-section, most of the common ones are. Typical conduits of noncircular cross-sections include heating and air conditioning ducts that are often of rectangular cross-sections. Normally the pressure difference between the inside and outside of these ducts is relatively small. Most of the basic principles involved are independent of the cross-sectional shape, although the flow details may depend on it. Unless otherwise specified, we will assume that the conduit is round.

The ***Reynolds number*** is an important dimensionless number in fluid mechanics

$$Re = \frac{\rho \cdot V \cdot D}{\mu} = \frac{V \cdot D}{\nu}. \qquad (9.1)$$

It is probably the most important fluid parameter when talking about viscous flow in pipes. It’s defined as the density times the average velocity times inside diameter of the pipe (if the pipe is circular) divided by the absolute viscosity of the fluid (or the average velocity times diameter divided by kinematic viscosity).

The Reynolds Number tells us what the flow is: laminar or turbulent. The Reynolds Number ranges for which laminar, transitional, or turbulent pipe flows are obtained cannot be precisely given. The transition from laminar to turbulent flow may occur at various Reynolds numbers, depending on how much the flow is disturbed by vibrations of the pipe, the roughness of the entrance region, and the like.

For general engineering purposes (i.e., without undue precautions to eliminate such disturbances), the following values are appropriate:

- if the Re < 2100 the flow is laminar;
- if the Re > 4000 the flow is turbulent.

In between these two numbers is a transition zone (critical region) where both laminar and turbulent flow can occur in an apparently random fashion.

For laminar flow in a pipe there is only one component of velocity, u. For turbulent flow, the predominant component of velocity is also along the pipe, but it is unsteady (random) and accompanied by random components normal to the pipe axis, *u*, *v*, *w*. Such motion in a typical flow occurs too fast for our eyes to follow.

$$Re = \frac{\rho \cdot V \cdot D}{\mu} = \frac{V \cdot D}{\nu}. \quad (9.2)$$

If we rewrite the Reynolds number using the fact that Q = V·A we can obtain

$$Re = \frac{V \cdot D}{\nu} = \frac{Q \cdot D}{\frac{\pi \cdot D^2}{4} \cdot \nu},$$

or

$$Re = \frac{4Q}{\pi \cdot D \cdot \nu}. \quad (9.3)$$

Using the fact that $\dot{m} = \rho \cdot Q$ we can also get

$$Re = \frac{4\dot{m}}{\pi \cdot D \cdot \mu}. \quad (9.4)$$

For noncircular ducts the Reynolds Number can be defined as

$$Re = \frac{V \cdot D_h}{\nu}, \quad (9.5)$$

where D_h is a hydraulic diameter, which can be defined as

$$D_h = \frac{4A}{P}, \quad (9.6)$$

where *A* is a cross-sectional area, *P* is wetted perimeter (the perimeter where the flowing fluid touches the wall of the duct).

Laminar and turbulent flows look different. The velocity profile of laminar flow is parabolic (Fig. 9.1a), whereas the velocity profile of turbulent flow is almost uniform over the central region of flow but very steep near the walls (Fig. 9.1b).

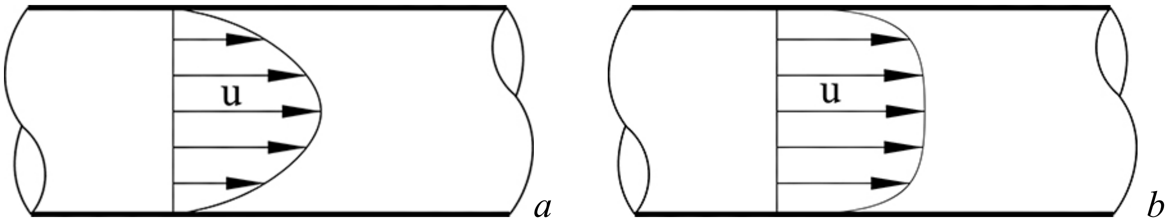


Fig. 9.1. Velocity profiles of laminar and turbulent flows

Because of the difference between profiles the relation between the average fluid velocity V and maximum velocity u can be expressed as:

- for the laminar flow

$$\frac{V}{u_{max}} = \frac{1}{2},$$

- for the turbulent flow

$$\frac{V}{u_{max}} \approx \frac{4}{5}.$$

Any fluid flowing in a pipe had to enter the pipe at some location. The region of flow near where the fluid enters the pipe is termed the ***entrance region***.

As it is shown in Fig. 9.2, the fluid typically enters the pipe with a nearly uniform velocity profile at section (1). As the fluid moves through the pipe, viscous effects cause it to stick to the pipe wall (the no-slip boundary condition). This is true whether the fluid is relatively inviscid air or a very viscous oil. Thus, a ***boundary layer*** in which viscous effects are important is produced along the pipe wall such that the initial velocity profile changes with distance along the pipe, *x*, until the fluid reaches the end of the entrance ***length, section*** (2), beyond which the velocity profile does not vary with *x*. The boundary layer has grown in thickness to completely fill the pipe. Viscous effects are of considerable importance within the boundary layer. For fluid outside the boundary layer [within the inviscid core surrounding the centerline from (1) to (2)], viscous effects are negligible.

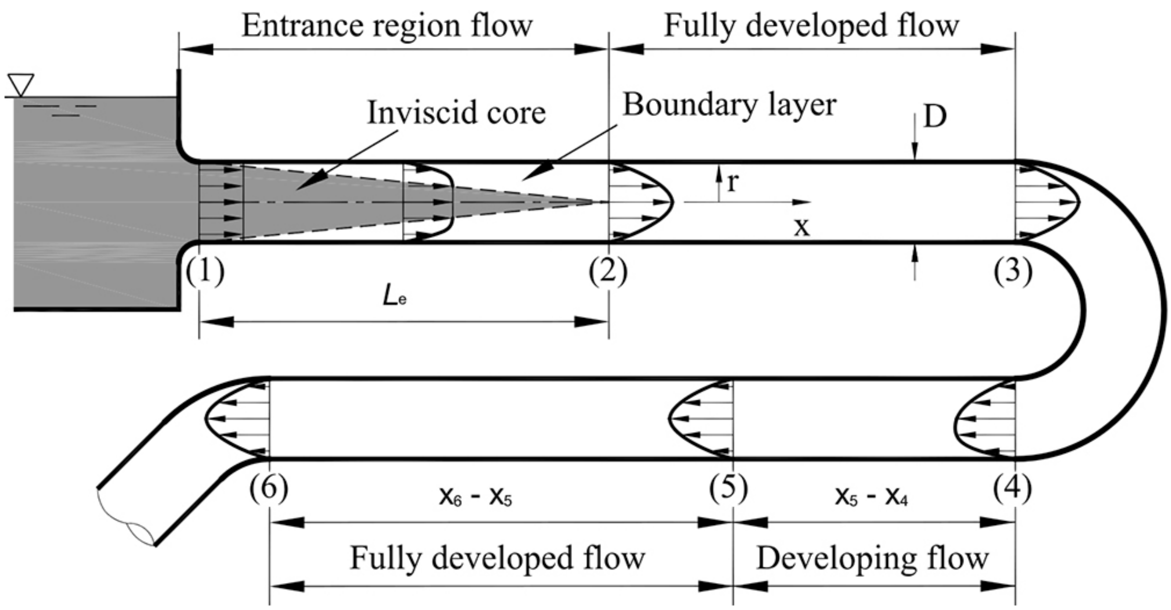


Fig. 9.2. Entrance region, developing flow, and fully developed flow in a pipe system

The shape of the velocity profile in the pipe depends on whether the flow is laminar or turbulent, as does the length of the entrance region, l_e. As with many other properties of pipe flow, the dimensionless ***entrance length***, l_e/D, correlates quite well with the Reynolds number. Typical entrance lengths are given by:

- for the laminar flow

$$\frac{l_e}{D} = 0.06\,Re\,,$$

- for the turbulent flow

$$\frac{l_e}{D} = 4.4\,(Re)^{1/6}\,.$$

For flows with very low Reynolds Number the entrance length can be quite short ($l_e = 0.6D$ if $Re = 10$), whereas for flows with large Reynolds Number it may take a length equal to many pipe diameters before the end of the entrance region is reached ($l_e = 120D$ if $Re = 2000$). For many practical engineering problems ($10^4 < Re < 10^5$), entrance region is $20D < l_e < 30D$.

Calculation of the velocity profile and pressure distribution within the entrance region is quite complex. Once the fluid reaches the end of the entrance region, section (2) of Fig. 9.2, the flow is simpler to describe because the velocity is a function of only the distance from the pipe centerline, r, and independent of x. This is true until the character of the pipe changes in some way, such as a change in diameter, or the fluid flows through a bend, valve, or some other component at section (3). The flow between (2) and (3) is termed ***fully developed flow***. Beyond the interruption of the fully developed flow [at section (4)], the flow gradually begins its return to its fully developed character [section (5)] and continues with this profile until the next pipe system component is reached [section (6)]. In many cases the pipe is long enough so that there is a considerable length of fully developed flow compared with the developing flow length [$(x_3 - x_2) >> le$ and $(x_6 - x_5) >> (x_5 - x_4)$]. In other cases, the distances between one component (bend, tee, valve, etc.) of the pipe system and the next component is so short that fully developed flow is never achieved.

Let's label parameters of the flow in sections 1 through 6: mass rate $\dot{m}_1 - \dot{m}_6$, flowrate $Q_1 - Q_6$, average velocity $V_1 - V_6$, pressure $P_1 - P_6$. If it is steady state due to conservation of mass we can have

$$\dot{m}_1 = \dot{m}_2 = \dot{m}_3 = \dot{m}_4 = \dot{m}_5 = \dot{m}_6 .$$

Flowrate $Q = \dot{m}/\rho$, if the flow is incompressible, we can have

$$Q_1 = Q_2 = Q_3 = Q_4 = Q_5 = Q_6 .$$

Average velocity $V = Q \cdot A$, if the cross-sectional area of the pipe is the same, we can have

$$V_1 = V_2 = V_3 = V_4 = V_5 = V_6 .$$

The last parameter is pressure. Die to the friction losses along the pipe the pressure is decreasing as it is shown in the Fig. 9.3,

$$P_1 > P_2 > P_3 > P_4 > P_5 > P_6 .$$

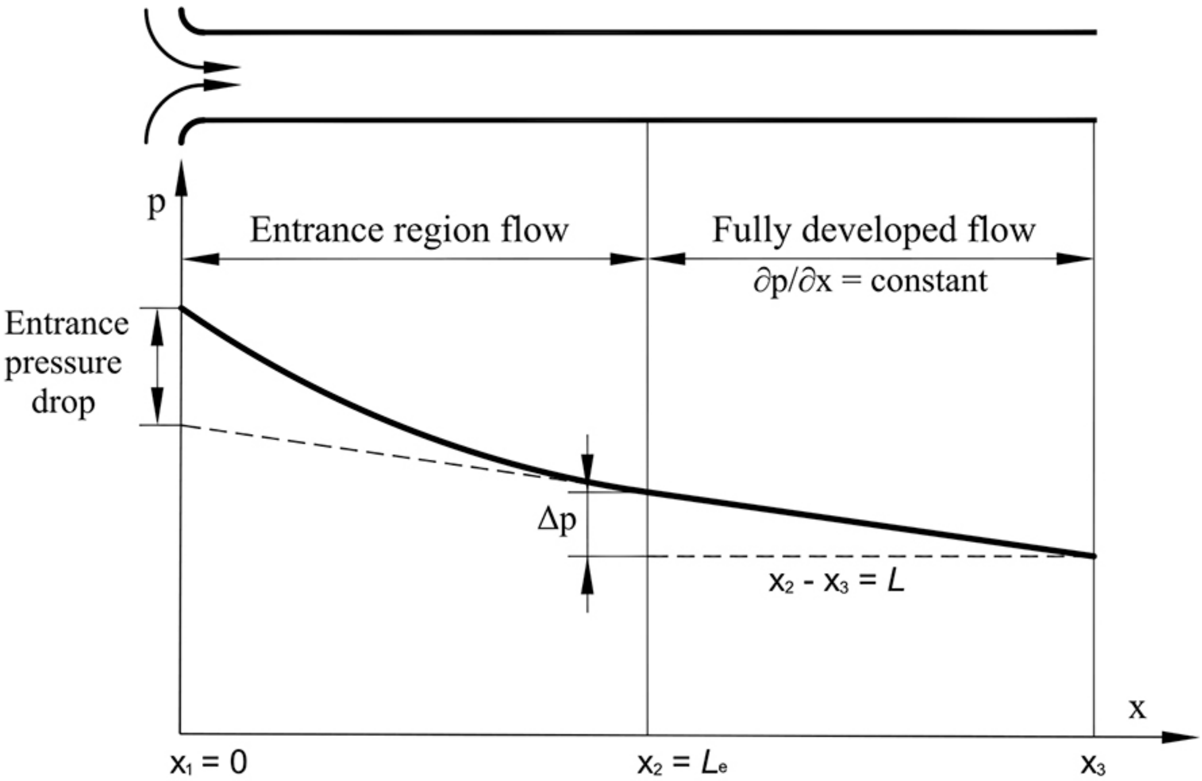


Fig. 9.3. Pressure distribution along a horizontal pipe

Laminar flow in pipes

We consider the fluid element at time t as is shown in Fig. 9.4. It is a circular cylinder of fluid of length l and radius r centered on the axis of a horizontal pipe of diameter D. Because the velocity is not uniform across the pipe, the initially flat ends of the cylinder of fluid at time t become distorted at time $t + \delta t$ when the fluid element has moved to its new location along the pipe as shown in the figure. If the flow is fully developed and steady, the distortion on each end of the fluid element is the same, and no part of the fluid experiences any acceleration as it flows.

Thus, every part of the fluid merely flows along its streamline parallel to the pipe walls with constant velocity, although neighboring particles have slightly different velocities. The velocity varies from one pathline to the next. This velocity variation, combined with the fluid viscosity, produces the shear stress.

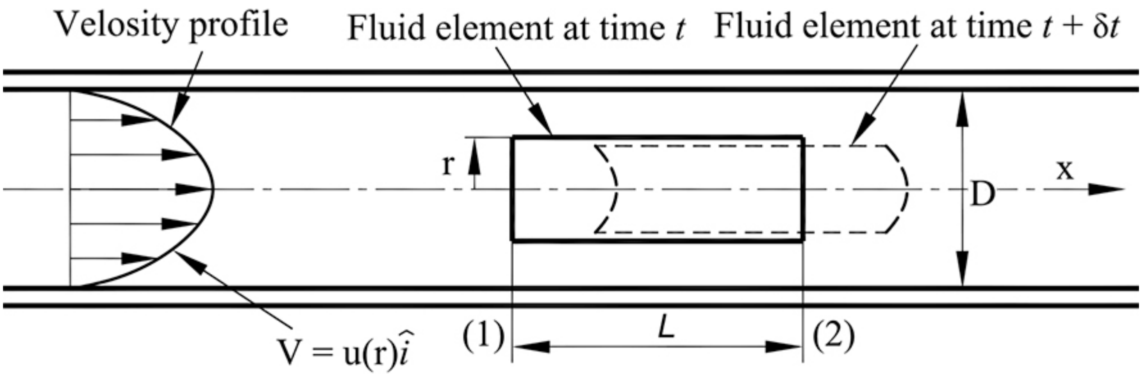


Fig. 9.4. Motion of a cylindrical fluid element within a pipe

We anticipate the fact that the pressure decreases in the direction of flow so that $\Delta p = p_1 - p_2 > 0$. A shear stress, acts on the surface of the cylinder of fluid.

We can isolate the cylinder of fluid as is shown in Fig. 9.5 and apply Newton's second law taking into account that there is no acceleration.

In this case the summation of forces, which are change in momentum is equal zero

$$\sum F = 0 \,.$$

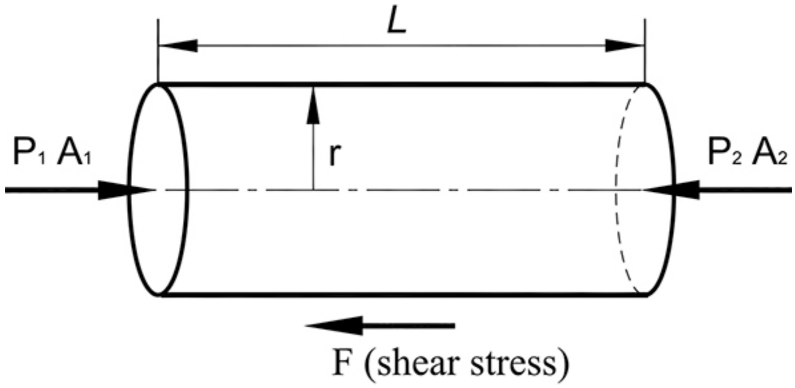


Fig. 9.5. Free-body diagram of a cylinder of fluid

Fully developed horizontal pipe flow is merely a balance between pressure and viscous forces – the pressure forces acting on the left and right ends of the cylinder and the shear stress acting on the lateral surface of the cylinder.

Area of an end of the cylinder is

$$A_1 = A_2 = \pi \cdot r^2 .$$

Area of the lateral surface of the cylinder is

$$A_l = 2\pi \cdot r \cdot l .$$

Thus, the force balance can be written as

$$P_1 \cdot \pi \cdot r^2 \; - P_2 \cdot \pi \cdot r^2 \; - \tau \cdot 2\pi \cdot r \cdot l = 0.$$

We can rearrange it to

$$\pi \cdot r^2 \cdot (P_1 - P_2) = \tau \cdot (2\pi \cdot r \cdot l).$$

If labeled $\Delta P = (P_1 - P_2)$, further rearrangement gives

$$\frac{\Delta P}{l} = \frac{2\tau}{r}. \tag{9.7}$$

Equation (9.7) represents the basic balance in forces needed to drive each fluid particle along the pipe with constant velocity. The left side of the equation is related to pressure forces and the right side is related to shear forces. So, we can conclude that for the small cylinder of fluid the net pressure force equals to the net shear force. The two balance each other.

We know that according to the Newton's Law of Viscosity for a laminar flow the shear stress can be written as

$$\tau = -\mu \cdot \frac{du}{dr}.$$

So, we can rewrite Equation (9.7) as

$$\frac{\Delta P}{l} = -\frac{2\mu}{r} \cdot \frac{du}{dr}.$$

Rearrangement gives us

$$\frac{du}{dr} = -\left(\frac{\Delta P}{2\mu \cdot l}\right) \cdot r.$$

Now we can integrate this equation to obtain

$$u = -\left(\frac{\Delta P}{2\mu \cdot l}\right) \cdot \frac{r^2}{2} + C,$$

or

$$u = -\left(\frac{\Delta P}{4\mu \cdot l}\right) \cdot r^2 + C, \tag{9.8}$$

where *C* is a constant.

Because the fluid is viscous it sticks to the pipe wall. In means that if $r = D/2$ (at the pipe wall) velocity is zero, $u = 0$. From the Equation (9.8) we can conclude that

$$C = \left(\frac{\Delta P}{4\mu \cdot l}\right) \cdot r^2,$$

or (using the fact that $r = D/2$)

$$C = \left(\frac{\Delta P}{16\mu \cdot l}\right) \cdot D^2. \tag{9.9}$$

Hence, using Equation (9.9) we can rewrite Equation (9.8) as

$$u = -\left(\frac{\Delta P}{4\mu \cdot l}\right) \cdot r^2 + \left(\frac{\Delta P}{16\mu \cdot l}\right) \cdot D^2.$$

Rearrangement allows to obtain

$$u = -\left(\frac{\Delta P \cdot D^2}{16\mu \cdot l}\right) \cdot \frac{4r^2}{D^2} + \left(\frac{\Delta P}{16\mu \cdot l}\right) \cdot D^2,$$

or

$$u = \left(\frac{\Delta P \cdot D^2}{16\mu \cdot l}\right) \cdot \left[1 - \left(\frac{2r}{D}\right)^2\right]. \tag{9.10}$$

If we put r = 0 (the center of the pipe) into Equation (9.10) we obtain the velocity at the pipe centerline

$$V_c = \frac{\Delta P \cdot D^2}{16\mu \cdot l}. \tag{9.11}$$

So, the Equation (9.10) can be rewritten as

$$u = V_c \cdot \left[1 - \left(\frac{2r}{D}\right)^2\right]. \tag{9.12}$$

This velocity profile, plotted in Fig. 9.6, is parabolic in the radial coordinate, *r*, has a maximum velocity, at the pipe centerline, and a minimum velocity (zero) at the pipe wall.

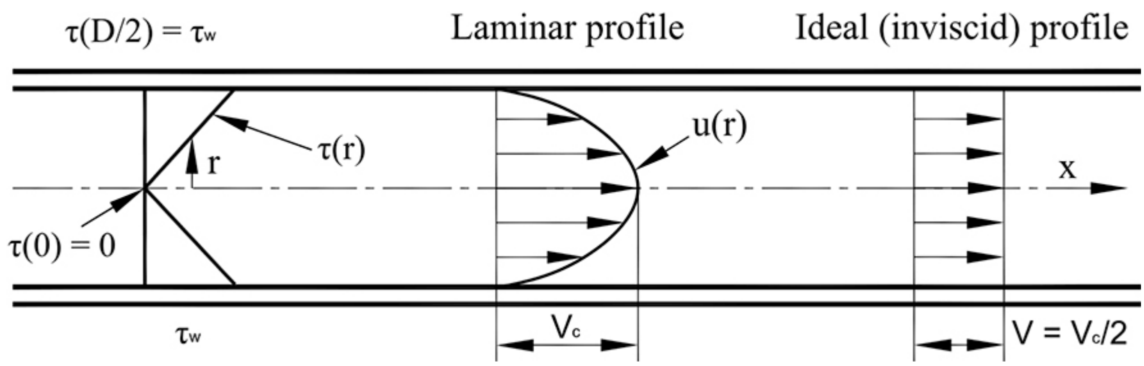


Fig. 9.6. Shear stress distribution within the fluid in a pipe (laminar or turbulent flow) and typical velocity profiles

The volume flowrate through the pipe can be obtained by integrating the velocity profile across the pipe.

$$Q = \int_r u \cdot dA. \tag{9.13}$$

Since the flow is axisymmetric about the centerline, the velocity is constant on small area elements consisting of rings of radius *r* and thickness *dr*, as shown in the Fig. 9.7.

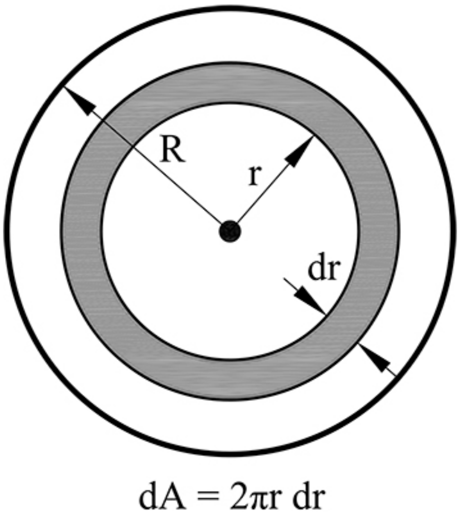


Fig. 9.7. Ring differential area element

The area of the element is circumference multiplied by thickness

$$dA = 2\pi \cdot r \cdot dr. \tag{9.14}$$

So, the integrating gives

$$Q = \int_r u \cdot dA = \int_{r=0}^{r=R} u\,(r) \cdot 2\pi \cdot r \cdot dr$$
$$= 2\pi \cdot V_c \cdot \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r \cdot dr.$$

or

$$Q = \frac{\pi \cdot R^2 \cdot V_c}{2}. \tag{9.15}$$

By definition, the average velocity is the flowrate divided by the cross-sectional area

$$V = \frac{Q}{\pi \cdot R^2}. \tag{9.16}$$

So that for this flow

$$V = \frac{\pi \cdot R^2 \cdot V_c}{2\pi \cdot R^2},$$

which means that the average velocity and the velocity at the pipe centerline are related as

$$V = \frac{1}{2} \cdot V_c. \tag{9.17}$$

or

$$V = \frac{\Delta P \cdot D^2}{32\mu \cdot l}. \tag{9.18}$$

As is indicated in Equation (9.17), the average velocity is one-half of the maximum velocity.

We can obtain by putting the Equation (9.11) into the Equation (9.15)

$$Q = \frac{\pi \cdot R^2}{2} \cdot \frac{\Delta P \cdot D^2}{16\mu \cdot l}.$$

or

$$Q = \frac{\pi \cdot D^4 \cdot \Delta P}{128 \cdot \mu \cdot l}. \tag{9.19}$$

Equation (9.19) is commonly referred to as ***Poiseuille's law***.

The above results confirm the following properties of laminar pipe flow. For a horizontal pipe the flowrate is (a) directly proportional to the pressure drop, (b) inversely proportional to the viscosity, (c) inversely proportional to the pipe length, and (d) proportional to the pipe diameter to the fourth power. With all other parameters fixed, an increase in diameter by a factor of 2 will increase the flowrate by a factor of $2^4 = 16$. It means that the flowrate is very strongly dependent on pipe size. A small error in pipe diameter can cause a relatively large error in flowrate.

Recall that all of these results are restricted to laminar flow (those with Reynolds numbers less than approximately 2100) in a horizontal pipe.

If we solve Equation (9.19) for ΔP we obtain

$$\Delta P = \frac{128 \cdot \mu \cdot l \cdot Q}{\pi \cdot D^4}, \tag{9.20}$$

or, if we put Equation (9.16) into Equation (9.20)

$$\Delta P = \frac{32 \cdot \mu \cdot l \cdot V}{D^2}. \tag{9.21}$$

We can divide Equation (9.21) by ${}^1\!/_2\, \rho V^2$, which is dynamic pressure

$$\frac{\Delta P}{{}^1\!/_2\, \rho \cdot V^2} = \frac{\frac{32 \cdot \mu \cdot l \cdot V}{D^2}}{{}^1\!/_2\, \rho \cdot V^2}.$$

It gives

$$\frac{\Delta P}{{}^1\!/_2\, \rho \cdot V^2} = 64 \left(\frac{\mu}{\rho \cdot V \cdot D}\right) \cdot \left(\frac{l}{D}\right).$$

Because $(\rho \cdot V \cdot D)/\mu$ is the Reynolds Number write

$$\Delta P = \left(\frac{64}{Re}\right) \cdot \left(\frac{l}{D}\right) \cdot \left(\frac{\rho \cdot V^2}{2}\right). \tag{9.22}$$

or

$$\Delta P = f \cdot \frac{l}{D} \cdot \frac{\rho \cdot V^2}{2}. \tag{9.23}$$

where f is a dimensionless ***Friction Factor*** (sometimes it is called the ***Darcy Friction Factor***) for a laminar fully developed pipe flow (obtained analytically)

$$f = \frac{64}{Re}. \tag{9.24}$$

The friction factor, or sometimes it is called the Darcy Friction Factor, is an invented factor, defined by Equation (9.23), which helps to calculate a pressure drop in a pipe (energy losses) due to friction.

For turbulent flow the dependence of the friction factor on the Reynolds number is much more complex than that given by Equation (9.24) for laminar flow. It will be discussed in detail later.

Now we can divide Equation (9.23) by $\rho \cdot g$ to get in meter, *m*, or feet, *ft*, units

$$h_l = \frac{\Delta P}{\gamma} = f \cdot \frac{l}{D} \cdot \frac{V^2}{2g}. \tag{9.25}$$

Equation (9.25) defines the head losses due to friction, h_l. To this end we consider the energy equation for incompressible, steady flow between two locations as is given in the Energy equation (Equation (7.23)) and transform it to

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_P = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_T + h_l + h_m. \tag{9.26}$$

where, h_P, head developed by the pump, h_T, head developed by the turbine (must be negative), h_l, head losses due to pipe friction (l stands for the length of a piece pipe), h_m, minor losses due to valves, elbows, fittings etc.

Equation (9.26) is often called ***Modified Bernoulli's equation***.

THEME 10. TURBULENT FLOW IN PIPES

Since turbulent pipe flow is actually more likely to occur than laminar flow in practical situations, it is necessary to obtain similar information for turbulent pipe flow. However, turbulent flow is a very complex process. Although a considerable amount of knowledge about the topic has been developed, the field of turbulent flow still remains the least understood area of fluid mechanics.

Consider a long section of pipe that is initially filled with a fluid at rest. As the valve is opened to start the flow, the flow velocity and, hence, the Reynolds number increase from zero (no flow) to their maximum steady-state flow values, as is shown in Fig. 10.1. Assume this transient process is slow enough so that unsteady effects are negligible (quasi-steady flow). For an initial time period the Reynolds number is small enough for laminar flow to occur. At some time, the Reynolds number reaches 2100, and the flow begins its transition to turbulent conditions. Intermittent spots or bursts of turbulence appear. As the Reynolds number is increased, the entire flow field becomes turbulent. The flow remains turbulent as long as the Reynolds number exceeds approximately 4000.

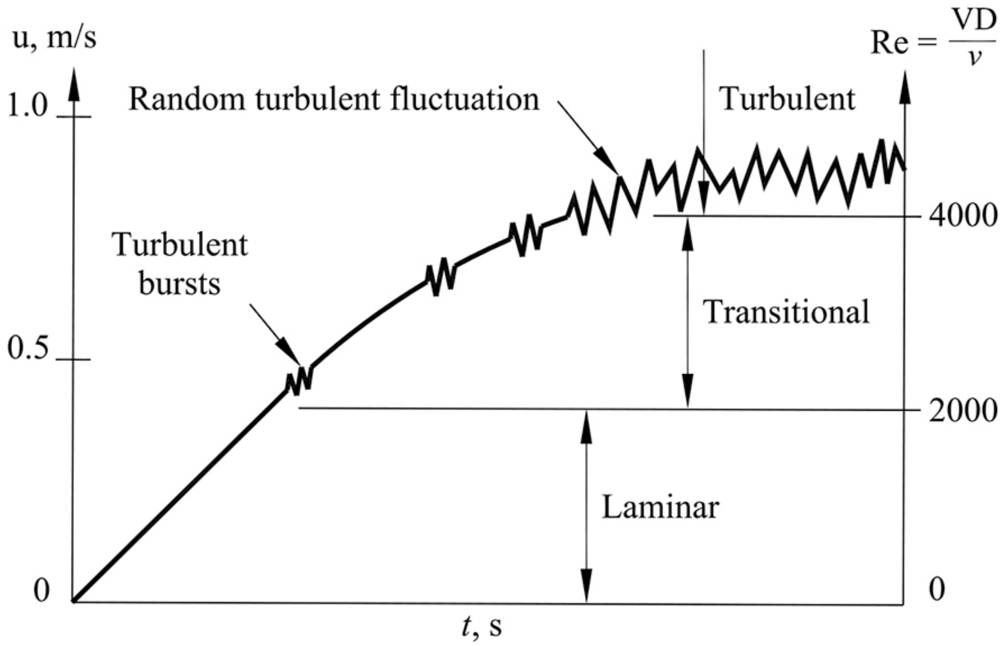


Fig. 10.1. Transition from laminar to turbulent flow in a pipe

A typical trace of the axial component of velocity measured at a given location in the flow (it is shown in Fig. 10.1 at Re > 4000). Its irregular, random nature is the distinguishing feature of turbulent flow. The character of many of the important properties of the flow (pressure drop, heat transfer, etc.) depends strongly on the existence and nature of the turbulent fluctuations or randomness indicated.

Without turbulence it would be virtually impossible to carry out life as we now know it. Mixing is one positive application of turbulence, but there are other situations where turbulent flow is desirable. To transfer the required heat between a solid and an adjacent fluid (such as in the cooling coils of an air conditioner or a boiler of a power plant) would require an enormously large heat exchanger if the flow were laminar. Similarly, the required mass transfer of a liquid state to a vapor state (such as is needed in the evaporated cooling system associated with sweating) would require very large surfaces if the fluid flowing past the surface were laminar rather than turbulent.

Turbulence is also of importance in the mixing of fluids. It is considerably easier to mix cream into a cup of coffee (turbulent flow) than to thoroughly mix two colors of a viscous paint (laminar flow).

In other situations, laminar (rather than turbulent) flow is desirable. The pressure drop in pipes (hence, the power requirements for pumping) can be considerably lower if the flow is laminar rather than turbulent.

The fundamental difference between laminar and turbulent flow lies in the chaotic, random behavior of the various fluid parameters. Such variations occur in the three components of velocity, the pressure, the shear stress, the temperature, and any other variable that has a field description.

Such flows can be described in terms of their mean values (denoted with an overbar) on which are superimposed the fluctuations (denoted with a prime).

The fluctuating part of the velocity, u', is that time-varying portion that differs from the average value

$$u = \overline{u} + u' \qquad or \quad u' = u - \overline{u}.$$

The structure and characteristics of turbulence may vary from one flow situation to another. For example, the *turbulence intensity* (or the level of the turbulence) may be larger in a very gusty wind than it is in a relatively steady (although turbulent) wind.

The larger the turbulence intensity, the larger the fluctuations of the velocity (and other flow parameters).

It is tempting to extend the concept of viscous shear stress for laminar flow ($\tau = \mu\, du/dy$) to that of turbulent flow by replacing u, the instantaneous velocity, by $\overline{u}$, the time-averaged velocity. However, numerous experimental and theoretical studies have shown that such an approach leads to completely incorrect results. That is, $\tau \neq$

$\mu d\bar{u}/dy$. The relationship between fluid motion and shear stress is very complex for turbulent flow.

Turbulent flow shear stress is larger than laminar flow shear stress because of the irregular, random motion. The shear stress is the sum of a laminar portion and a turbulent portion.

Although the relative magnitude of the laminar portion, τ_{lam}, compared to turbulent portion, τ_{turb}, is a complex function dependent on the specific flow involved, typical measurements indicate the structure shown in Fig. 10.2 (the shear stress is proportional to the distance from the centerline of the pipe). In a very narrow region near the wall (the viscous sublayer), the laminar shear stress is dominant. Away from the wall (in the outer layer) the turbulent portion of the shear stress is dominant. The transition between these two regions occurs in the overlap layer. The corresponding typical velocity profile is shown in Fig. 10.2b (the scale of the sketches shown in Fig. 10.2 is not necessarily correct).

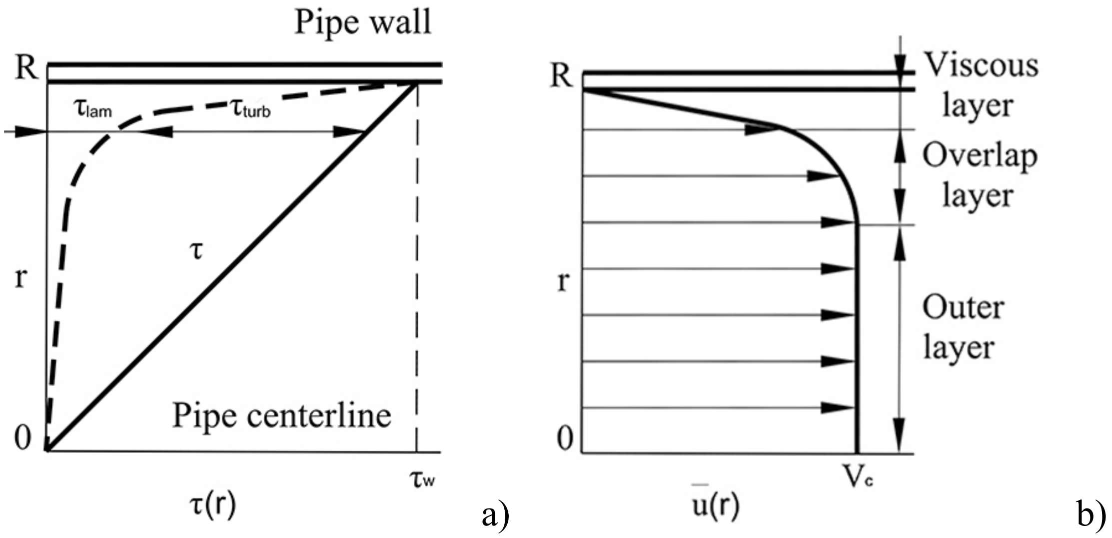


Fig. 10.2. Structure of turbulent flow in a pipe. (a) Shear stress. (b) Average velocity

Most turbulent pipe flow analyses are based on experimental data and semi-empirical formulas. These data are expressed conveniently in dimensionless form.

It is often necessary to determine the head loss, h_L, that occurs in a pipe flow so that the Energy equation, Equation (7.23), can be used in the analysis of pipe flow problems. The overall head loss for the pipe system consists of the head loss due to viscous effects in the straight pipes, termed the ***major loss*** and denoted h_l, and the head loss in the various pipe components, termed the ***minor loss*** and denoted h_m, that is

$$h_L = h_l + h_m.$$

The head loss designations of "major" and "minor" do not necessarily reflect the relative importance of each type of loss. For a pipe system that contains many components and a relatively short length of pipe, the minor loss may actually be larger than the major loss.

Major losses (friction losses)

Major losses are losses due to friction in the flow.

Although the pressure drop for laminar pipe flow is found to be independent of the roughness of the pipe, it is necessary to include this parameter when considering turbulent flow. There was determined dimensionless ***relative roughness*** of pipe material as

$$Relative\ roughness = \frac{\varepsilon}{D},$$

where, ε, is a measure of the roughness of the pipe wall, D, pipe diameter.

For determining the pressure drop for turbulent flow in horizontal pipe it is convenient to use the same equation as for laminar flow (Equation (9.25)). But it is much more difficult to determine the friction factor, f, for turbulent flow.

For turbulent flow, the functional dependence of the friction factor on the Reynolds number and the ***relative roughness***, is rather complex one that cannot, as yet, be obtained from a theoretical analysis. The results are obtained from an exhaustive set of experiments and usually presented in terms of a curve-fitting formula or the equivalent graphical form.

Much of this information is a result of experiments conducted by J. Nikuradse in 1933 and amplified by many others since then. One difficulty lies in the determination of the roughness of the pipe. Nikuradse used artificially roughened pipes produced by gluing sand grains of known size onto pipe walls to produce pipes with sandpaper-type surfaces. The pressure drop needed to produce a desired flowrate was measured, and the data were converted into the friction factor for the corresponding Reynolds number and relative roughness. The tests were repeated numerous times for a wide range of Re and ε/D to determine the dependence.

In commercially available pipes the roughness is not as uniform and well defined as in the artificially roughened pipes used by Nikuradse. However, it is possible to obtain a measure of the ***effective relative roughness*** of typical pipes and thus to obtain the friction factor. Typical roughness values for various pipe surfaces are given in Table 10.1.

It is important to observe that the values of relative roughness given pertain to new, clean pipes. After considerable use, most pipes (because of a buildup of corrosion or scale) may have a relative roughness that is considerably larger (perhaps by an order of magnitude) than that given.

Fig. 10.3 shows the functional dependence of f on Re and ε/D and is called the ***Moody Chart (Moody Diagram)***. It should be noted that the values of ε/D do not necessarily correspond to the actual values obtained by a microscopic determination of the average height of the roughness of the surface. They do, however, provide the correct correlation for $f = f(Re, \varepsilon/D)$.

Table 10.1. Equivalent Roughness for New Pipes

Pipe material	Equivalent Roughness, e	
	Millimeters	**Feet**
Riveted steel	0.9÷9.0	0.003÷0.03
Concrete	0.3÷3.0	0.001÷0.01
Wood stave	0.18÷0.9	0.0006÷0.003
Cast iron	0.26	0.00085
Galvanized iron	0.15	0.0005
Commercial steel or wrought iron	0.045	0.00015
Drawn tubing	0.0015	0.000005
Plastic, glass	0.0 (smooth)	0.0 (smooth)

The following characteristics are observed from the data of Fig. 10.3. For laminar flow, which is independent of relative roughness $f = 64/Re$.

For turbulent flows with very large Reynolds numbers, $f = f(\varepsilon/D)$, which, as shown by the figure in the margin, is independent of the Reynolds number. For such flows, commonly termed ***completely turbulent flow*** (or ***wholly turbulent flow***), the laminar sublayer is so thin (its thickness decreases with increasing Re) that the surface roughness completely dominates the character of the flow near the wall. Hence, the pressure drop required is a result of an inertia-dominated turbulent shear stress rather than the viscosity-dominated laminar shear stress normally found in the viscous sublayer.

For flows with moderate values of Re, the friction factor is indeed dependent on both the Reynolds number and relative roughness, $f = f(Re, \varepsilon/D)$.

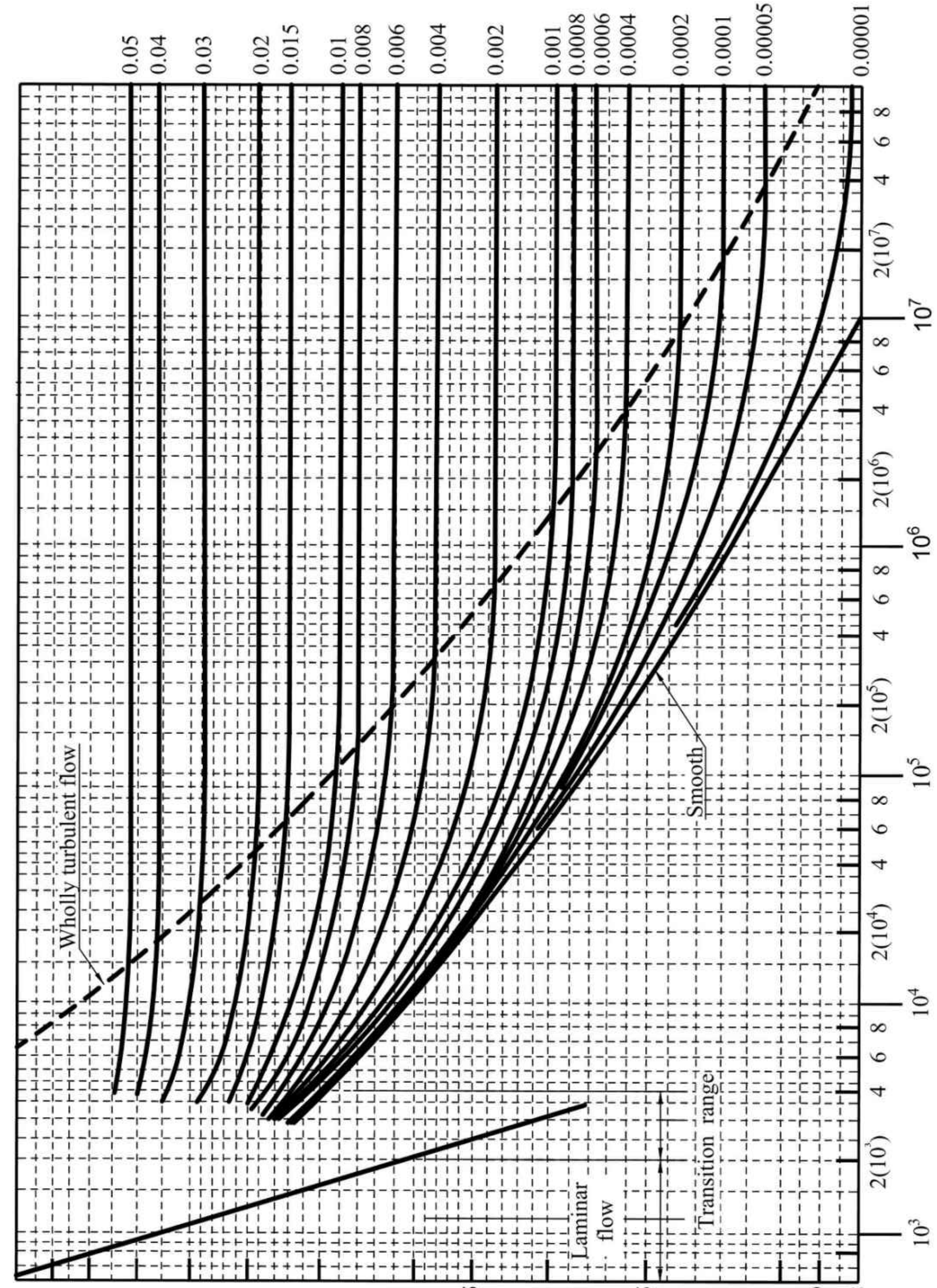


Fig. 10.3. Friction factor as a function of Reynolds number and relative roughness for round pipes – the Moody Chart

The gap in the figure for which no values of f are given (the $2100 < Re < 4000$ range) is a result of the fact that the flow in this transition range may be laminar or turbulent (or an unsteady mix of both) depending on the specific circumstances involved.

It is important to note that even for smooth pipes, $\varepsilon = 0$, the friction factor is not zero. It means that there is a head loss in any pipe, no matter how smooth the surface is made. This is a result of the no-slip boundary condition that requires any fluid to stick to any solid surface it flows over. Such pipes are called ***hydraulically smooth***.

The Moody Chart covers an extremely wide range in flow parameters. The nonlaminar region covers more than four orders of magnitude in Reynolds Number from $Re = 4 \times 10^3$ to $Re = 10^8$. Generally, the Moody chart is universally valid for all steady, fully developed, incompressible pipe flows.

There are four regions of the Moody diagram (from the left to the right) – laminar region, critical region, transition region, completely turbulent region.

For laminar region f depends on Re only.

For transition region f depends on Re and ε/D.

For completely turbulent region f depends on ε/D only.

The following equation is valid for the entire nonlaminar range of the Moody chart

$$\frac{1}{\sqrt{f}} = -1.8log\left[\left(\frac{\varepsilon/D}{3.7}\right)^{1.11} + \frac{69}{Re}\right]. \qquad (10.1)$$

In fact, the Moody chart is a graphical representation of this equation, which is an empirical fit of the pipe flow pressure drop data. Equation (10.1) is called the ***Modified Colebrook formula***.

Because of various inherent inaccuracies involved (uncertainty in the relative roughness, uncertainty in the experimental data used to produce the Moody chart, etc.), solving pipe flow problems using Moddy Chart or Colebrook Formula is usually not very accurate. As a rule of thumb, a 10% accuracy is the best expected.

Minor losses

Minor losses are losses due to fittings in the pipe (valves, elbows, tees etc.).

The head loss associated with flow through a valve is a common ***minor loss***. The flow resistance or head loss through the valve may be a significant portion of the resistance in the system. In fact, with the valve closed, the resistance to the flow is infinite – the fluid cannot flow. With the valve wide open the extra resistance due to the presence of the valve may or may not be negligible.

Theoretical analysis to predict the details of such flows to obtain the head loss for these components is not, as yet, possible. Thus, the head loss information for essentially all components is given in dimensionless form and based on experimental data. The most common method used to determine these head losses or pressure drops is to specify the ***minor loss coefficient***, which is defined as

$$K_L = \frac{h_m}{\left(V^2/_{2g}\right)} = \frac{\Delta P}{\frac{1}{2}\rho \cdot V^2}. \qquad (10.2)$$

So, the minor losses can be defined as

$$\Delta P = K_L \frac{\rho \cdot V^2}{2}, \qquad (10.3)$$

or

$$h_m = K_L \frac{V^2}{2g}. \qquad (10.4)$$

The actual value of K_L is strongly dependent on the geometry of the component considered. For most flows the loss coefficient is independent of the Reynolds number.

Typical values of ***minor loss coefficient*** for pipe components are given in Table 10.2. These typical components are designed more for ease of manufacturing and costs than for reduction of the head losses that they produce.

Minor losses are sometimes given in terms of an ***equivalent length***, l_{eq}. In this terminology, the head loss through a component is given in terms of the equivalent length of pipe that would produce the same head loss as the component. That is,

$$h_m = f \cdot \frac{l_{eq}}{D} \cdot \frac{V^2}{2g}. \qquad (10.5)$$

where D and f are based on the pipe containing the component. The head loss of the pipe system is the same as that produced in a straight pipe whose length is equal to the pipes of the original system plus the sum of the additional equivalent lengths of all of the components of the system.

Table 10.2. Minor Loss Coefficients for Pipe Components

Component	**K_L**
Elbows	
Regular 90^0, flanged	0.3
Regular 90^0, threaded	1.5
Long radius 90^0, flanged	0.2
Long radius 90^0, threaded	0.7
Long radius 45^0, flanged	0.2
Regular 45^0, threaded	0.4
180^0 return bends	
180^0 return bend, flanged	0.2
180^0 return bend, threaded	1.5
Tees	
Line flow, flanged	0.2
Line flow, threaded	0.9
Branch flow, flanged	1.0
Branch flow, threaded	2.0
Unions	
Union, threaded	0.08
Valves	
Globe, fully open	10.0
Angle, fully open	2.0
Gate, fully open	0.5
Gate, ¼ closed	0.26
Gate, ½ closed	2.1
Gate, ¾ closed	17.0
Swing check, forward flow	2.0
Swing check, backward flow	∞
Ball valve, fully open	0.05
Ball valve, $^1/_3$ closed	5.5
Ball valve, $^2/_3$ closed	210

Pipe flow problems can be categorized by what parameters are given and what is to be calculated (Table 10.3).

Table 10.3. Categorize of pipe flow problems

Variable	**Type I**	**Type II**	**Type III**
	a. Fluid		
Density	Given	Given	Given
Viscosity	Given	Given	Given
	b. Pipe		
Diameter	Given	Given	Determine
Length	Given	Given	Given
Roughness	Given	Given	Given
	c. Flow		
Flowrate or Average Velocity	Given	Determine	Given
	d. Pressure		
Pressure Drop or Head Loss	Determine	Given	Given

In a Type I problem we specify the desired flowrate or average velocity and determine the necessary pressure difference or head loss.

In a Type II problem, we specify the applied driving pressure (or, alternatively, the head loss) and determine the flowrate.

In a Type III problem, we specify the pressure drop and the flowrate and determine the diameter of the pipe needed.

THEME 11. DIMENSIONALITY ANALYSIS AND SIMILITUDE

The solution to many problems is achieved through the use of a combination of theoretical and numerical analysis and experimental data.

An obvious goal of any experiment is to make the results as widely applicable as possible. To achieve this end, the concept of ***similitude*** is often used so that measurements made on one system (for example, in the laboratory) can be used to describe the behavior of other similar systems (outside the laboratory). The laboratory systems are usually thought of as ***models*** and are used to study the phenomenon of interest under carefully controlled conditions. It is necessary to establish the relationship between the laboratory model and the "other" system.

Data are best presented in *dimensionless* form. Experiments which might result in tables of output, or even multiple volumes of tables, might be reduced to a single set of curves – or even a single curve – when suitably nondimensionalized. The technique for doing this is *dimensional analysis*.

Consider a smooth sphere of diameter D that is put in flow of fluid of density ρ and viscosity μ (it can be a wind tunnel). Velocity of the flow is V. There is a drag force F_D on it because the fluid is moving towards the sphere as it is shown in Fig. 11.1.

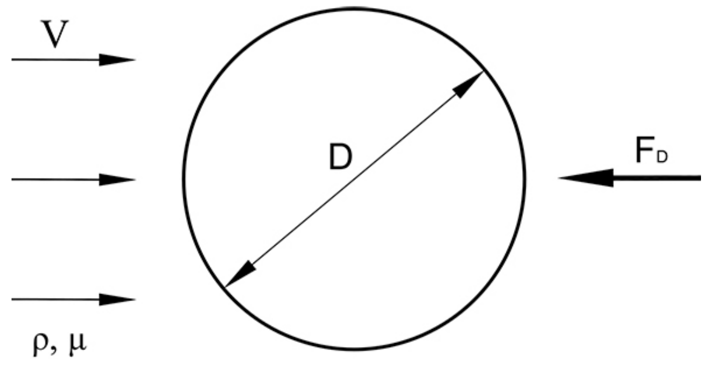


Fig. 11.1. Drag force on a cylinder

The drag force depends on diameter of the sphere, characteristics of the fluid and its velocity

$$F_D = f(D, V, \rho, \mu). \tag{11.1}$$

So, there are 5 variables: F_D, D, V, ρ and μ. We need to find out experimentally the dependance of the drag force on different things, such as flows of different velocity, fluids of different characteristics (ρ and μ) and different diameters of the sphere. To obtain this data we must run huge number of experiments. It will take a lot of time and result in hundreds of diagrams.

Dimensional analysis can help to do it way much faster. The data can be represented in dimensionless coordinates. We can define dimensionless Drag Coefficient as

$$C_D = \frac{F_D}{\rho \cdot V^2 \cdot D^2}.$$

We can run just one experiment and then we can plot the results of the experiment in *Drag Coefficient*, K_D, and *Reynolds Number*, *Re*, dimensionless coordinates as it is shown in Fig. 44. These coordinates are the important dimensionless parameters for this problem. The single graph created in this way can be valid for different velocity, different diameters of the sphere and different characteristics of the fluid. The whole crux of the matter is if we can't find these important dimensionless parameters, then it's not going to work like that. The key is how to find them. In order to do this we have to fulfill a special procedure, which is ***dimensional analysis***. When doing this analysis, we would start with Equation (11.1) and end up with the next one

$$F_D \frac{F_D}{\rho \cdot V^2 \cdot D^2} = f\left(\frac{\rho \cdot V \cdot D}{\mu}\right). \tag{11.2}$$

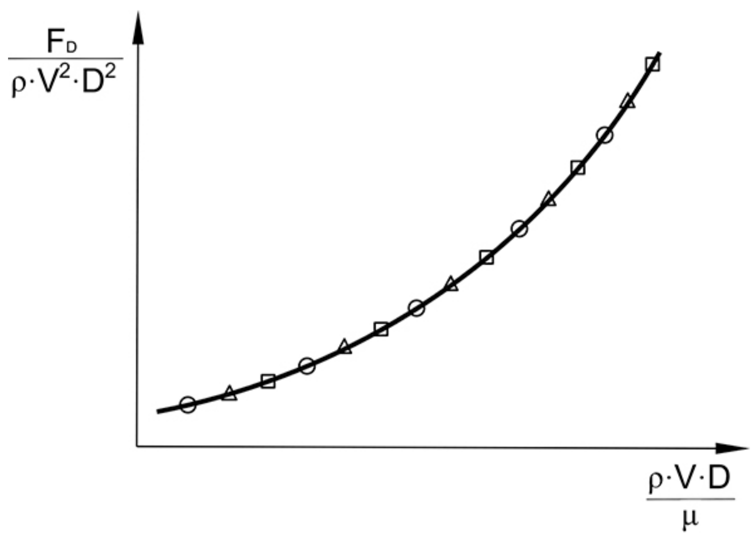


Fig. 11.2. Representation of dependence of drag force on different parameters in dimensionless coordinates

Π-Theorem (officially "The Buckingham Π-Theorem")

Π-Theorem is a way to identify the important dimensionless parameters in s problem. There are several steps to do it.

Step 1. Select pertinent variables (there are n of those).

Step 2. Wright the functional relation

Step 3. Select repeating variables (there are 3 sub-steps)

a) don't select a dependent variable;

b) variables should contain all m major dimensions (M, L ant T);

c) do not select dimensionless variables as repeating.

Step 4. The number of Π-parameters are $n - m$.

Step 5. Write the Π-terms by combining the repeating variables with each of the remaining variables tacked on the end.

Step 6. Solve the equations from step 5.

Step 7. Write the functional relationship.

$$\Pi_1 = f(\Pi_2, \Pi_3, etc.).$$

When it's completed, we end up with functional relationship between the Π-parameters.

Dimensions of fluid-mechanics properties is shown in Table 1.1.

For the problem of finding dependance of the drag force on different characteristics the 7 steps should be the next.

Step 1. There are five pertinent variables: F_D, D, V, ρ and μ (n = 5).

Step 2. The functional relation between them is $F_D = f(D, V, \rho, \mu)$.

Step 3. a) Drag force F_D is a dependent variable, so it's not to be selected as repeating variable.

b) When selecting repeating variables, we can check presence in them of all m major dimensions (M, L ant T) in Table 1.1.

To do this we can write down all variables with their dimensions

$$D = [L], V = \left[\frac{L}{T}\right], \rho = \left[\frac{M}{L^3}\right], \mu = \left[\frac{M}{L \cdot T}\right], F_D = \left[\frac{M \cdot L}{T^2}\right].$$

c) There are no dimensionless variables to select.

We can select as repeating variables ρ, V and D. We can see all m major dimensions (M, L ant T) at least ones among the repeating variable's dimensions.

Step 4. The number of important dimensionless Π-parameters that would characterize this problem are

$$\#\Pi = n - m = 5 - 3 = 2.$$

Step 5. We can write the two Π-terms. We repeat three chosen repeating variables ρ, V and D raising them to a, b and c powers respectively and tacking remaining variables F_D and μ on the end

$$\Pi_1 = \rho^a \cdot V^b \cdot D^c \cdot F_D, \qquad \Pi_2 = \rho^a \cdot V^b \cdot D^c \cdot \mu.$$

Step 6. Solving the equations from step 5 gives

$$\Pi_1 = \left[\frac{M}{L^3}\right]^a \cdot \left[\frac{L}{T}\right]^b \cdot [L]^c \cdot \left[\frac{M \cdot L}{T^2}\right], \qquad \Pi_2 = \left[\frac{M}{L^3}\right]^a \cdot \left[\frac{L}{T}\right]^b \cdot [L]^c \cdot \left[\frac{M}{L \cdot T}\right].$$

$$\Pi_1 = M^{a+1} \cdot L^{-3a+b+c+1} \cdot T^{-b-2}, \qquad \Pi_2 = M^{a+1} \cdot L^{-3a+b+c-1} \cdot T^{-b-1}.$$

So

$$a + 1 = 0 \quad \rightarrow \quad a = -1, \qquad a + 1 = 0 \quad \rightarrow \quad a = -1.$$

$$-b - 2 = 0 \quad \rightarrow \quad b = -2, \qquad -b - 1 = 0 \quad \rightarrow \quad b = -1.$$

$$-3a + b + c + 1 = 0 \quad \rightarrow \quad c = -1, \qquad -3a + b + c - 1 = 0 \quad \rightarrow \quad c = -1.$$

It means that

$$\Pi_1 = \rho^{-1} \cdot V^{-2} \cdot D^{-2} \cdot F_D, \qquad \Pi_2 = \rho^{-1} \cdot V^{-1} \cdot D^{-1} \cdot \mu.$$

Then we obtain two dimensionless Π-terms

$$\Pi_1 = \frac{F_D}{\rho \cdot V^2 \cdot D^2}, \qquad \Pi_2 = \frac{\mu}{\rho \cdot V \cdot D}.$$

Now, we can use Π_1 and Π_2 as the coordinates to plot the results of the experiments. It will be a single line as it is shown in Fig. 11.2.

We could select other repeating variables, for example V, D and μ. After completing the analysis and plotting the results of the experiments we will obtain a different line because of e different function. However, it will meet our goals.

Sometimes we can look at the governing equation of some situation and from that determine what the important Π-parameters are.

Consider two horizontal plates with spacing h between them as it is shown in the Fig. 11.3. Fluid with kinematic viscosity ν is between them. The bottom plate is fixed and the top plate is given a velocity U. ν

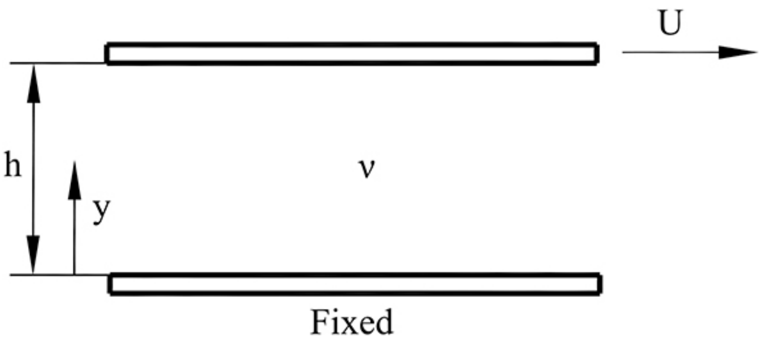


Fig. 11.3. Relative movement of horizontal plates with a fluid between them

The equation that describes this situation is a variation of the Navier-Stokes equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{11.3}$$

We are going to normalize this equation (make the equation dimensionless).

Sometimes if we have the differential equation, we might be able to find out what the important Π-parameters are for a particular problem. But it is not easy.

We want this equation to be in dimensionless variables. To do this, we have to get rid of dimension of velocity u, dimension of distance y and dimension of time t. We want to have dimensionless velocity, dimensionless distance and dimensionless time.

Let's multiply and divide the left-hand side and right-hand side of the Equation (11.3) by U. We obtain

$$\frac{U}{U} \cdot \frac{\partial u}{\partial t} = \nu \cdot \frac{U}{U} \cdot \frac{\partial^2 u}{\partial y^2}.$$

Because U is constant, we can bring the U in denominators inside the partial sines. So, we have

$$U \cdot \frac{\partial \left(\frac{u}{U}\right)}{\partial t} = \nu \cdot U \cdot \frac{\partial^2 \left(\frac{u}{U}\right)}{\partial y^2}.$$

Now, we can cancel U and do the next labeling

$$u^* = \frac{u}{U}.$$

In this case the velocity u^* is dimensionless. Then we can write

$$\frac{\partial u^*}{\partial t} = \nu \frac{\partial^2 u^*}{\partial y^2}.$$

We can do the same thing with the distance y. Let's do the labeling

$$y^* = \frac{y}{h}.$$

In this case the distance y^* becomes dimensionless. We can write then

$$\frac{\partial u^*}{\partial t} = \nu \cdot \frac{h^2}{h^2} \cdot \frac{\partial^2 u^*}{\partial y^2}.$$

We can bring the h^2 in the numerator in the right-hand side of the equation inside the partial sine in the denominator. So, we have

$$\frac{\partial u^*}{\partial t} = \frac{\nu}{h^2} \cdot \frac{\partial^2 u^*}{\partial \left(\frac{y}{h}\right)^2} = \frac{\nu}{h^2} \cdot \frac{\partial^2 u^*}{\partial y^{*2}}. \tag{11.4}$$

To make time dimensionless, we have to find something which has time in it. We can do it this way

$$t' = \frac{h}{U} = \frac{[L]}{\frac{[L]}{[T]}} = [T].$$

Then, we can wright dimensionless time as

$$t^* = \frac{t}{t'} = \frac{t}{\frac{h}{U}}.$$

Let's multiply the left-hand side of the Equation (11.4) by two reciprocal ratios U/h and h/U . It does not change the equation

$$\frac{U}{h}\cdot\frac{h}{U}\cdot\frac{\partial u^*}{\partial t}=\frac{\nu}{h^2}\cdot\frac{\partial^2 u^*}{\partial y^{*2}}.$$

Now, we bring the ratio h/U in the left-hand side of the equation, which has dimension of time, inside the partial sine in the denominator

$$\frac{U}{h}\cdot\frac{\partial u^*}{\partial\left(\frac{t}{h/U}\right)}=\frac{\nu}{h^2}\cdot\frac{\partial^2 u^*}{\partial y^{*2}}.$$

Finally, we obtain ***dimensionless governing partial differential equation***

$$\frac{\partial u^*}{\partial t^*}=\frac{\nu}{h\cdot U}\cdot\frac{\partial^2 u^*}{\partial y^{*2}}$$

It means that the important Π-parameter is

$$\Pi=\frac{\nu}{h\cdot U}.$$

Sometimes by non-dimensionalizing a basic equation we can obtain an important dimensionless parameter.

Common fluid mechanics dimensionless groups and their physical meanings

The list of variables that commonly arise in fluid mechanics problems can look like this:

- acceleration of gravity, g;
- bulk modulus, E_v;
- characteristic length, l;
- density, ρ;
- frequency of oscillating flow, ω;
- pressure, p (or Δp);
- speed of sound, c;
- surface tension, σ;
- velocity, V;
- viscosity, μ.

The list is obviously not exhaustive but does indicate a broad range of variables likely to be found in a typical problem. Fortunately, not all of these variables would be encountered in all problems. However, when combinations of these variables are present, it is standard practice to combine them into some of the common dimensionless groups. The groups are given in Table 11.1. These combinations appear so frequently that special names are associated with them, as indicated in the table.

It is also often possible to provide a physical interpretation to the dimensionless groups which can be helpful in assessing their influence in a particular application.

Reynolds Number. In most fluid flow problems, there will be a characteristic length L, and a velocity, V, as well as the fluid properties of density, ρ, and viscosity, μ, which are relevant variables in the problem. Thus, with these variables the Reynolds number

$$\mathrm{Re} = \frac{\rho \cdot V \cdot L}{\mu}.$$

To rearrange Re we can multiply it by V/V, L/L and $1/_{L/L}$. It does not change the equation. We get

$$\mathrm{Re} = \frac{\rho \cdot V \cdot L}{\mu} \cdot \frac{V}{V} \cdot \frac{L}{L} \cdot \frac{1}{L/_L} = \frac{\rho \cdot V^2 \cdot L^2}{\left(\frac{\mu \cdot V}{L}\right) \cdot L^2}.$$

In this equation:

- $\rho \cdot V^2$ is proportional to $\rho \cdot V^2/_2$ which is dynamic pressure and L^2 is an area;
- $\mu \cdot V/_L$ is shear stress ($\tau = \mu \cdot du/_{dy}$) and L^2 is also an area.

So, physical interpretation of the Reynolds Number is inertia force (dynamic pressure multiplied by area) divided by viscous force (viscous stress multiplied by area).

The Reynolds number is a measure of the ratio of the inertia force on an element of fluid to the viscous force on an element. When these two types of forces are important in a given problem, the Reynolds Number will play an important role. However, if the Reynolds number is very small (Re < 1) this is an indication that the viscous forces are dominant in the problem, and it may be possible to neglect the inertial effects; that is, the density of the fluid will not be an important variable. Flows at very small Reynolds numbers are commonly referred to as "creeping flows".

Conversely, for large Reynolds Number flows, viscous effects are small relative to inertial effects and for these cases it may be possible to neglect the effect of viscosity and consider the problem as one involving a "nonviscous" fluid.

Table 11.1. Some Common Variables and Dimensionless Groups in Fluid Mechanics

Name	Dimensionless Groups	Interpretation	Types of Application
Reynolds Number, Re	$\frac{\rho V l}{\mu}$	$\frac{inertia\ force}{viscous\ force}$	Generally, of importance in all types of fluid dynamics problems
Mach Number, Ma*	$\frac{V}{c}$	$\frac{inertia\ force}{compressibility\ force}$	Problems in which compressibility of the fluid is important
Froude Number, Fr	$\frac{V}{\sqrt{gl}}$	$\frac{inertia\ force}{gravitational\ force}$	Flow with a free surface
Euler Number, Eu	$\frac{p}{\rho V^2}$	$\frac{pressure\ force}{inertia\ force}$	Problems in which pressure, or pressure differences, are of interest
Weber Number, We	$\frac{\rho V^2 l}{\sigma}$	$\frac{inertia\ force}{surface\ tension\ force}$	Problems in which surface tension is important
Cauchy Number, Ca*	$\frac{\rho V^2}{E_v}$	$\frac{inertia\ force}{compressibility\ force}$	Problems in which compressibility of the fluid is important
Strouhal Number, St	$\frac{\omega l}{V}$	$\frac{inertia\ (local)\ force}{vinertia\ (convective)\ force}$	Unsteady flow with a characteristic frequency of oscillation

* The Cauchy number and the Mach number are related, and either can be used as an index of the relative effects of inertia and compressibility.

Mach number and Cauchy number. Either number (but not both) may be used in problems in which fluid compressibility is important. Both numbers can be interpreted as representing an index of the ratio of inertial forces to compressibility forces. When the Mach number is relatively small (say, less than 0.3), the inertial forces induced by the fluid motion are not sufficiently large to cause a significant change in the fluid density, and in this case the compressibility of the fluid can be neglected. The Mach number is the more commonly used parameter in compressible flow problems, particularly in the fields of gas dynamics and aerodynamics.

Froude number. It is distinguished from the other dimensionless groups in Table 11.1 is that it contains the acceleration of gravity, g. The acceleration of gravity becomes an important variable in a fluid dynamics problem in which the fluid weight is an important force. As discussed, the Froude number is a measure of the ratio of the inertia force on an element of fluid to the weight of the element. It will generally be important in problems involving flows with free surfaces since gravity principally affects this type of flow. Typical problems would include the study of the flow of water around ships (with the resulting wave action) or flow through rivers or open conduits.

It is to be noted that the Froude number is also commonly defined as the square of the Froude Number listed in Table 11.1.

Euler number. It can be interpreted as a measure of the ratio of pressure forces to inertial forces, where p is some characteristic pressure in the flow field. Very often the Euler Number is written in terms of a pressure difference, Δp, so that $Eu = \Delta p/\rho \cdot V^2$. Also, this combination expressed as $\Delta p \big/ \frac{1}{2}\rho \cdot V^2$ is called the *pressure coefficient*. Some form of the Euler number would normally be used in problems in which pressure or the pressure difference between two points is an important variable.

For problems in which cavitation is of concern, the dimensionless group $(p_r - p_v) \big/ \frac{1}{2}\rho \cdot V^2$ is commonly used, where, p_v, is the vapor pressure and, p_r, is some reference pressure. Although this dimensionless group has the same form as the Euler Number, it is generally referred to as the cavitation number.

Weber number. It may be important in problems in which there is an interface between two fluids. In this situation the surface tension may play an important role in the phenomenon of interest. The Weber Number can be thought of as an index of the inertial force to the surface tension force acting on a fluid element. Common examples of problems in which this parameter may be important include the flow of thin films of liquid, or the formation of droplets or bubbles. Clearly, not all problems involving flows with an interface will require the inclusion of surface tension. The flow of water in a river is not affected significantly by surface tension, since inertial and gravitational effects are dominant (We < 1).

However, as discussed in a later section, for river models (which may have small depths) caution is required so that surface tension does not become important in the model, whereas it is not important in the actual river.

Strouhal number. It is a dimensionless parameter that is likely to be important in unsteady, oscillating flow problems in which the frequency of the oscillation is ω. It represents a measure of the ratio of inertial forces due to the unsteadiness of the flow (local acceleration) to the inertial forces due to changes in velocity from point to point in the flow field (convective acceleration). This type of unsteady flow may develop when a fluid flows past a solid body (such as a wire or cable) placed in the moving stream. For example, in a certain Reynolds Number range, a periodic flow will develop downstream from a cylinder placed in a moving fluid due to a regular pattern of vortices that are shed from the body. When the frequency is in the audible range, a sound can be heard and the bodies appear to "sing."

The most dramatic evidence of this phenomenon occurred in 1940 with the collapse of the Tacoma Narrows Bridge. The shedding frequency of the vortices coincided with the natural frequency of the bridge, thereby setting up a resonant condition that eventually led to the collapse of the bridge.

Similarity studies

A *model* is a representation of a physical system that may be used to predict the behavior of the system in some desired respect. The physical system for which the predictions are to be made is called the *prototype*.

Although mathematical or computer models may also conform to this definition, our interest will be in physical models, that is, models that resemble the prototype but are generally of a different size, may involve different fluids, and often operate under different conditions (pressures, velocities, etc.).

Usually, a model is smaller than the prototype. Therefore, it is more easily handled in the laboratory and less expensive to construct and operate than a large prototype.

Occasionally, if the prototype is very small, it may be advantageous to have a model that is larger than the prototype so that it can be more easily studied.

We may also wish to examine a priori the effect of possible design changes that are proposed for a hydraulic structure or fluid flow system.

Similitude is the theory and art of prediction prototype performance from model observation.

Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype.

Instead of complete similarity, the engineering literature speaks of particular types of similarity, the most common being geometric, kinematic and dynamic. Let us consider each separately.

Geometric similarity

Geometric similarity concerns the length dimension, *L*, and must be ensured before any sensible model testing can proceed. A formal definition is as follows:

A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratio.

So, the next requirements must be met:

- *a model and a prototype must be of the same shape;*
- *all linear dimensions of the model must be related to the corresponding dimensions of the prototype by a constant scale factor;*
- *all angles are preserved in geometric similarity;*
- *orientations of model and prototype with respect to the surroundings must be identical.*

Kinematic similarity

Kinematic similarity requires that the model and prototype have the same length-scale ratio and the same time-scale ratio. The result is that the velocity-scale ratio will be the same for both.

So, the next requirements must be met:

- *velocity at corresponding points in the two flows must be in the same direction and must be related by a constant scale factor in magnitude;*
- *flow regimes must be the same (laminar, turbulent, compressible, ect.).*

Dynamic similarity

Dynamic similarity exists when the model and the prototype have the same length scale ratio, time-scale ratio, and force-scale (or mass-scale) ratio.

It means that:

- *identical kinds of forces at corresponding points in the flow must be parallel and related by a constant scale factor.*

THEME 12. COMPRESSIBLE FLOW

Compressible flow is the flow where compressibility effects are important in fluids (compressible flow is very mathematically oriented).

We usually treat the liquids as incompressible (water, oil, ect.). We often treat gases as incompressible, especially air. But air can be considered incompressible only if the Mach Number is less than about 0.3.

Equations that are important in compressible flow.

Ideal gas (or perfect gas) is a gas with constant specific heat.

The equation of state for an ideal gas is

$$\frac{P}{\rho} = R \cdot T. \tag{12.1}$$

where, P, absolute pressure T, absolute temperature, R, gas constant.

Gas constant represents a constant for each distinct ideal gas or mixture of ideal gases, where

$$R = \frac{\lambda}{M_{gas}}. \tag{12.2}$$

where, λ, is the universal gas constant and M_{gas}, is the molecular weight of the ideal gas or
gas mixture.

When knowing the pressure and temperature of a gas, we can estimate its density. Nonideal gas state equations are beyond the scope of this.

For an ideal gas, internal energy, is part of the stored energy of the gas and is considered to be a function of temperature only. So, the change in internal energy

$$du = C_V \cdot dT. \tag{12.3}$$

where, C_V, specific heat at constant volume.

Change in ***enthalpy***

$$dh = C_P \cdot dT. \tag{12.4}$$

where, C_P, specific heat at constant pressure.

Enthalpy can be expressed as combination of internal energy, and pressure energy

$$h = u + \frac{P}{\rho} = u + P \cdot V. \tag{12.5}$$

C_V and C_P specific are functions only of temperature and the next equation is valid

$$C_P - C_V = R. \tag{12.6}$$

Difference between two points in internal energy and ***enthalpy*** can be expressed as

$$u_2 - u_1 = C_V(T_2 - T_1), \tag{12.7}$$

$$h_2 - h_1 = C_P(T_2 - T_1). \tag{12.8}$$

If the *specific heat ratio*, *k*, is defined as

$$K = \frac{C_P}{C_V}, \tag{12.9}$$

we can express specific heats as following

$$C_V = \frac{R}{K - 1}, \tag{12.10}$$

$$C_P = \frac{K \cdot R}{K - 1}. \tag{12.11}$$

For compressible flows, changes in the thermodynamic property ***entropy***, *S*, are important. For any pure substance including ideal gases, the "first *TdS* equation" is

$$TdS = du + PdV, \tag{12.12}$$

or in another form

$$TdS = dh - VdP. \tag{12.13}$$

Solving for gives the "second *TdS* equation"

$$dS = \frac{du}{T} + \frac{PdV}{T} = C_V \frac{dT}{T} + R \frac{dV}{V}, \tag{12.14}$$

or in another form

$$dS = \frac{dh}{T} + \frac{VdP}{T} = C_P \frac{dT}{T} - R \frac{dV}{V}. \tag{12.15}$$

If C_V and C_P are assumed to be constant for a given gas, Equations (12.14) and (12.15) can be integrated to get

$$S_2 - S_1 = C_V \cdot ln \frac{T_2}{T_1} + R \cdot ln \frac{V_2}{V_1}, \tag{12.16}$$

$$S_2 - S_1 = C_P \cdot ln \frac{T_2}{T_1} - R \cdot ln \frac{P_2}{P_1}. \tag{12.17}$$

The *P* and *T* are absolute pressure and temperature.

The second law of thermodynamics requires that the ***adiabatic*** and frictionless flow of any fluid results in dS = 0 or $S_2 = S_1$. Constant entropy flow is called ***isentropic*** flow. For the isentropic flow of an ideal gas with constant C_V and C_P we get from Equations (12.16) and (12.17)

$$C_V \cdot ln \frac{T_2}{T_1} + R \cdot ln \frac{\rho_1}{\rho_2} = C_P \cdot ln \frac{T_2}{T_1} - R \cdot ln \frac{P_2}{P_1} = 0. \tag{12.18}$$

By combining Equation (12.18) with Equations (12.10) and (12.11) we obtain

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{K}{K-1}} = \left(\frac{\rho_2}{\rho_1}\right)^{K}. \tag{12.19}$$

From Equation (12.19) we can conclude that

$$\frac{P}{\rho^K} = P \cdot \rho^{-K} = const. \tag{12.20}$$

Speed of Sound in a compressible fluid (perfect gas)

Consider a wave of sound which is moving in a region where the air is still (the velocity of the air is zero, $V = 0$). The density of the air is ρ, the pressure is P (Fig. 12.1).

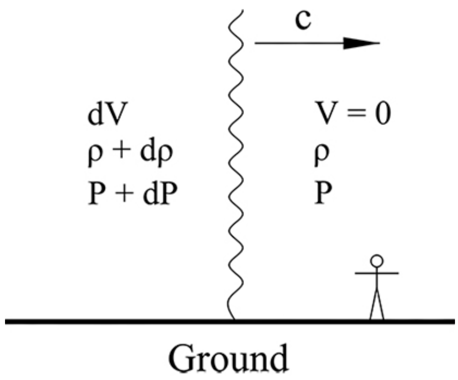


Fig. 12.1. Wave of sound

The observer is staying on the ground and the wave passes him. On the other side of the sound wave the parameters of the air change. They are: velocity is dV, the density is $\rho + d\rho$, the pressure is $P + dP$.

For the observer this is an unsteady process. When the sound wave passed it's quiet again and the air parameters become as they were before. To make it a steady process, we attach the observer to the wave (Fig. 12.2) and outline a control volume with a control surface.

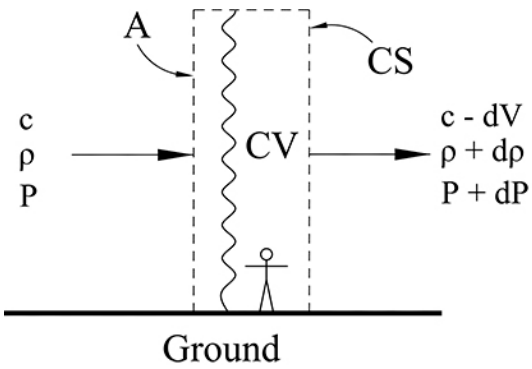


Fig. 12.2. Wave of sound

So, the observer is "sitting on the wave" and see the air approaching him at the speed of sound with parameters ρ and P. Parameters of air living the control volume are: velocity, $c - dV$, density, $\rho + d\rho$, pressure, $P + dP$.

Sound wave is a very week wave. There is a small differential change in pressure, velocity and density. To investigate it we apply continuity and momentum.

Conservation of mass

Steady state means that what comes in the control volume from one side goes out the other side, $\dot{m}_{in} = \dot{m}_{out}$. When the continuity equation is applied to the flow through this control volume, the result is

$$\rho \cdot A \cdot c = (\rho + d\rho)(c - dV)(A). \tag{12.21}$$

After cancelling A and rearrangement we obtain

$$\rho \cdot c = \rho \cdot c + c \cdot d\rho - \rho \cdot dV - d\rho \cdot dV. \tag{12.22}$$

We can ignore the higher order of differentials, $d\rho \cdot dV$, and get

$$0 = c \cdot d\rho - \rho \cdot dV. \tag{12.23}$$

x Momentum equation

On the left-hand side of this equation is summation of the forces acting on the control surface. Pressure times area gives us pressure force that are positive to the right, negative to the left. Pressure force always points into the control volume (compressing fluids). On the right-hand side the momentum flux. Mass flow $\dot{m}_{in} = \rho \cdot A \cdot c$, times velocity coming in, $\dot{m}_{in} \cdot c$, (minus sign because the velocity vector and the area vector point in opposite directions) plus mass flow times velocity going out, $\dot{m}_{out} \cdot (c - dV)$, (plus sign because the velocity vector and the area vector point in the same direction). So

$$+PA - (P + dP)(A) = -\rho \cdot A \cdot c^2 + \rho \cdot A \cdot c(c - dV). \tag{12.24}$$

The area cancels and we can solve it for the pressure change

$$dP = \rho \cdot c \cdot dV. \tag{12.25}$$

Taking into account Equation (12.23) we can write

$$dV = c\frac{d\rho}{\rho}. \tag{12.26}$$

So, we can get from Equation (12.25)

$$dP = c^2 \cdot d\rho, \tag{12.27}$$

or

$$c = \sqrt{\frac{dP}{d\rho}}. \tag{12.28}$$

It is valid for constant entropy (isentropic flow $S_1 = S_2$)

We are interested in finding $dP/d\rho$. According to Equation (12.20) we can write

$$P = (constant) \cdot \rho^k,$$

then

$$\frac{dP}{d\rho} = (constant) \cdot k \cdot \rho^{-k}.$$

But Equation (12.20) says that

$$(constant) = \frac{P}{\rho^k}$$

So

$$\frac{dP}{d\rho} = \frac{P}{\rho^k} \cdot k \cdot \rho^{-k} = \frac{P}{\rho} \cdot k.$$

Using Equation (12.1) we can write

$$\frac{dP}{d\rho} = k \cdot R \cdot T.$$

Finally, from Equation (12.28) we can write the equation for speed of sound

$$c = \sqrt{k \cdot R \cdot T}. \tag{12.29}$$

Consider flow from point (1) to point (2) in a flow field. Consider point (2) to be a stagnation point. Stagnation point is a point where the velocity comes to zero (fluid is brought to rest adiabatically, no heat transfer).

Equation for energy of fluid for this flow can be written as (while ignoring the change in elevation *z*)

$$C_P \cdot T + \frac{V^2}{2} = C_P \cdot T_0 + \frac{V_o^2}{2}. \quad (12.30)$$

where, $C_P \cdot T$, enthalpy at point (1), $C_P \cdot T_0$, enthalpy at stagnation point (2), $V^2/2$, kinetic energy at point (1), $V_0^2/2$, kinetic energy at stagnation point (2).

Velocity at stagnation point is zero by definition, so we can cancel the last term

$$C_P \cdot T + \frac{V^2}{2} = C_P \cdot T_0. \quad (12.31)$$

Now, we are going to solve it for the stagnation point. We can rewrite Equation (12.31) to get

$$\frac{T_0}{T} = \frac{V^2}{2C_P \cdot T} + 1. \quad (12.32)$$

But for a perfect gas

$$C_P \cdot T = \left[\frac{k \cdot R}{k-1}\right] \cdot T = \frac{C^2}{k-1}. \quad (12.33)$$

So

$$\frac{T_0}{T} = \frac{(k-1) \cdot V^2}{2c^2} + 1, \quad (12.34)$$

or

$$\frac{T_0}{T} = \frac{k-1}{2} \cdot M^2 + 1. \quad (12.35)$$

This is valid for an adiabatic process. Now we show where C_P in Equation (12.33) came from. We know from Equations (12.6) and (12.9) that $R = C_P - C_V$ and $K = C_P/C_V$. We can divide Equation (12.6) by C_P and use Equation (12.9) to get

$$\frac{R}{C_P} = 1 - \frac{C_V}{C_P} = 1 - \frac{1}{k} = \frac{k-1}{k}.$$

So, we put in Equation (12.33) C_P as

$$C_P = \frac{k \cdot R}{k-1}.$$

Now, using Equation (12.19) for an isentropic process (subindex 2 means stagnation point) we can combine it with Equation (12.35) to get

$$\frac{P_0}{P} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}, \tag{12.36}$$

and

$$\frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{1}{k-1}}. \tag{12.37}$$

If we want to find the stagnation temperature, we can use Equation (12.35), stagnation pressure – Equation (12.36), stagnation density – Equation (12.37). The characteristics with subindex (0) refer to stagnation point, the characteristics with no subindex refer to any other point in a flow field.

We can rewrite Equations (12.35), (12.36) and (12.37) for k = 1.4 (*Air*, N_2, O_2, *CO, NO*)

$$\frac{T_0}{T} = 1 + 0.2M^2, \tag{12.38}$$

$$\frac{P_0}{P} = (1 + 0.2M^2)^{3.5}, \tag{12.39}$$

$$\frac{\rho_0}{\rho} = (1 + 0.2M^2)^{2.5}. \tag{12.40}$$

For gases with $k = 1.4$ we can use Table A.1 (appendix A) instead of Equations (12.38), (12.39) and (12.40) to find unknown parameters of compressible flow. We have to know the Mach Number for it. Subindex (0) in this table refer to stagnation point, the characteristics with no subindex refer to any other point in a flow field. A^* stands for the area

At a location where M = 1.0 (sonic speed) for gases with k = 1.4 we have

$$\frac{P_0^*}{P_0} = \left(\frac{2}{k+2}\right)^{\frac{k}{k-1}} = 0.5283. \qquad (12.41a)$$

$$\frac{T_0^*}{T_0} = \frac{2}{k+1} = 0.8333. \qquad (12.41b)$$

$$\frac{\rho_0^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} = 0.6339. \qquad (12.41c)$$

Consider a duct in which the area varies (Fig. 12.3).

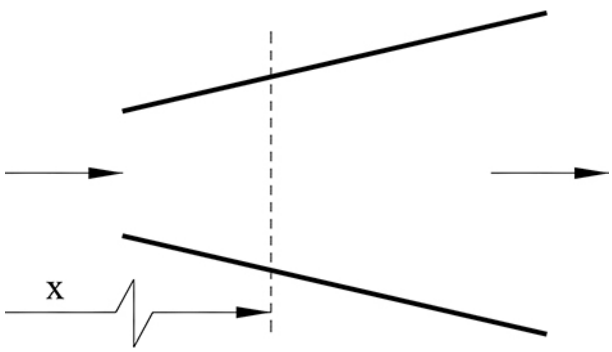


Fig. 12.3. A duct in which the area varies

Recall the Euler Equation in differential form, which is

$$VdV + \frac{dP}{\rho} = 0,$$

and continuity equation

$$\rho \cdot A \cdot V = constant.$$

Let's take a differential of the product

$$\rho \cdot A \cdot dV + \rho \cdot V \cdot dA + A \cdot V \cdot d\rho = 0.$$

Divide it through by $\rho \cdot A \cdot V$ to get

$$\frac{dV}{V} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0. \tag{12.42}$$

Then, we can use the equation of speed of sound

$$c = \sqrt{\frac{dP}{\rho}},$$

or

$$c^2 = \frac{dP}{\rho},$$

and put it into the Euler Equation to obtain

$$VdV + c^2 \frac{d\rho}{\rho} = 0. \tag{12.43}$$

We can solve Equation (12.43) for $d\rho/\rho$ and substitute it in Equation (12.42) to get

$$\frac{dV}{V} + \frac{dA}{A} - \frac{V \cdot dV}{c^2} = 0. \tag{12.44}$$

After multiplying by A/dV and canceling we obtain

$$\frac{A}{V} + \frac{dA}{dV} - \frac{A \cdot V}{c^2} = 0. \tag{12.45}$$

Solving for dA/dV gives

$$\frac{dA}{dV} = -\frac{A}{V} + \frac{A \cdot V}{c^2} = \frac{A}{V}\left(\frac{V^2}{c^2} - 1\right),$$

or

$$\frac{dA}{dV} = \frac{A}{V}(M^2 - 1) \qquad (12.46).$$

If the flow is subsonic ($M < 1$) then ($M^2 - 1$) is negative, so, the change in area with respect to velocity, dA/dV, is also negative. It means that if area of the duct decreases, the velocity gets larger (Fig. 12.4a). So, to get larger speed we must have a converging duct.

If the flow is supersonic ($M > 1$) then ($M^2 - 1$) is positive, so, the change in area with respect to velocity, dA/dV, is also positive. It means that if area of the duct decreases, the velocity gets smaller (Fig. 12.4b). So, to get larger speed we must have a diverging duct.

If we have sonic speed ($M = 1$) then

$$M^2 - 1 = 0,$$

and

$$\frac{dA}{dV} = 0,$$

or

$$dA = 0.$$

It means that in order to obtain sonic flow in a changing area duct we have to go through a local minimum of the area with $dA = 0$. Area must not change with x (Fig. 12.4c).

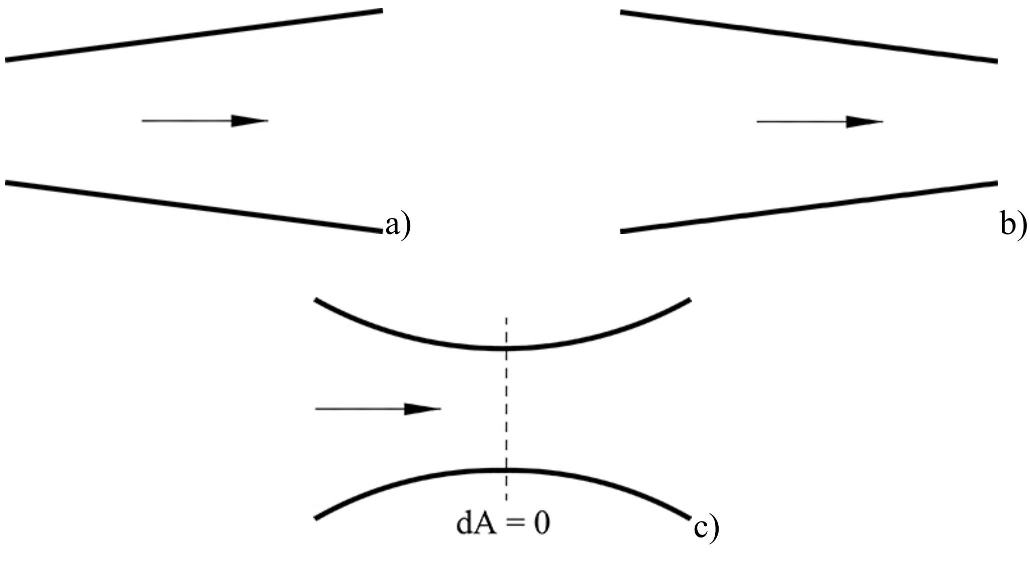


Fig. 12.4. Changing area ducts

Consider a large reservoir where fluid is at rest, the velocity of fluid is zero (Fig 12.5). If we want to construct, for example, a supersonic wind tunnel we need to accelerate the fluid from zero (inside the reservoir) to supersonic speed (in the nozzle).

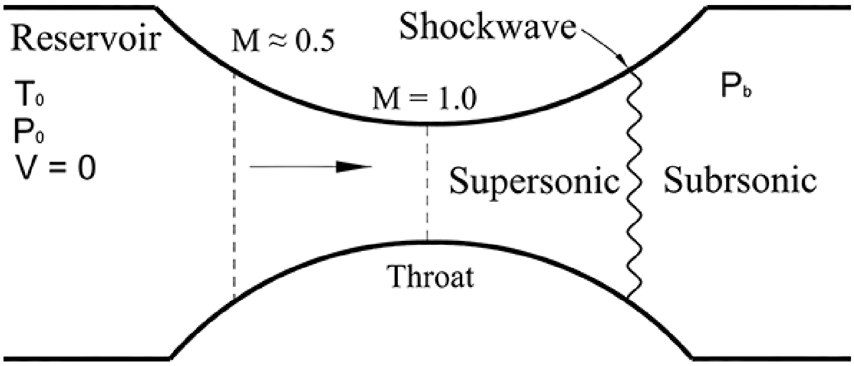


Fig. 12.5. Converging diverging nozzle

If V = 0, inside the reservoir we have stagnation temperature T_0 and stagnation pressure P_0. We start off to be subsonic. In order to make the velocity get larger we have to make the area get smaller. So, we begin with a converging nozzle. We have to continue with converging nozzle until the speed of the flow become sonic (M = 1.0). It must be the throat of the nozzle. Now, if we want the flow to go faster and become supersonic, we have to get the area bigger (down from the throat). Then the flow can reach M > 1.0. After that, we end up with the exit area and back pressure region P_b.

There could be a shock wave in the diverging part of the nozzle. In this case we would have supersonic speed before the shock wave and subsonic speed after it.

If the flow doesn't reach M = 1.0 at the throat area (reaches, for example M = 0.9) then it will not reach supersonic speed in diverging part of the nozzle. In this case after the throat area the speed starts to become smaller.

THEME 13. ISENTROPIC FLOW IN A CONVERGING NOZZLE

Consider a large reservoir with stagnation pressure and temperature and consider a nozzle that ends up with an exit (exit pressure P_e) and back pressure (P_b) area (Fig. 13.1).

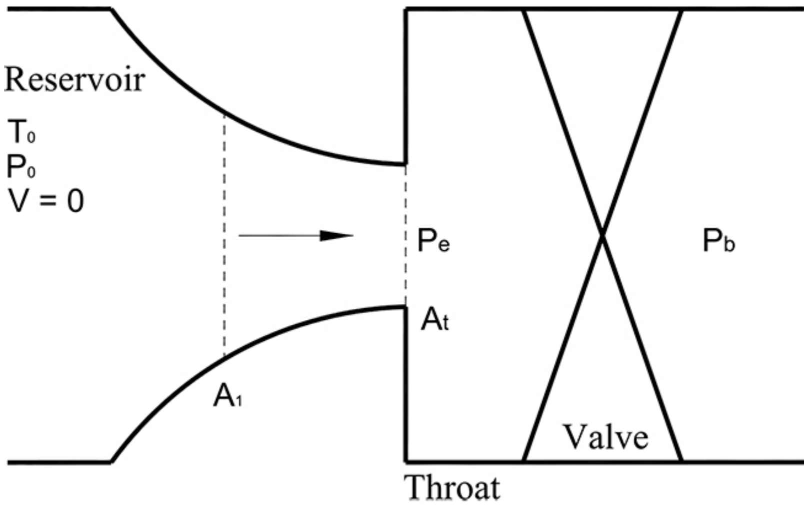


Fig. 13.1. Converging nozzle

It's not possible to have supersonic flow in converging nozzle. But it's possible to reach sonic speed (M = 1.0) at the exit of the nozzle under some conditions.

If the back pressure is the same as stagnation pressure, $P_b = P_0$, (the valve in the Fig. 13.1 is closed) there is no flow in the nozzle. If the valve is partly open the velocity becomes greater than zero and the back pressure drops. The fluid starts to accelerate through the nozzle. The ratio P/P_0 changes along the nozzle as it's shown in the Fig. 13.2.

If the valve is open more the velocity gets bigger and pressure drops more. The lower is the back pressure the larger is the velocity and the lower is the line in the figure.

At some point the valve is open enough to create velocity high enough to reach M = 1.0 at the exit. If the valve continues to get more open the back pressure is reduced even more. But the exit pressure does not change. No more flow goes through the exit. Now we get an expansion wave outside the nozzle. The lower is the back pressure the bigger is the expansion wave and nothing happens in the nozzle or before the nozzle.

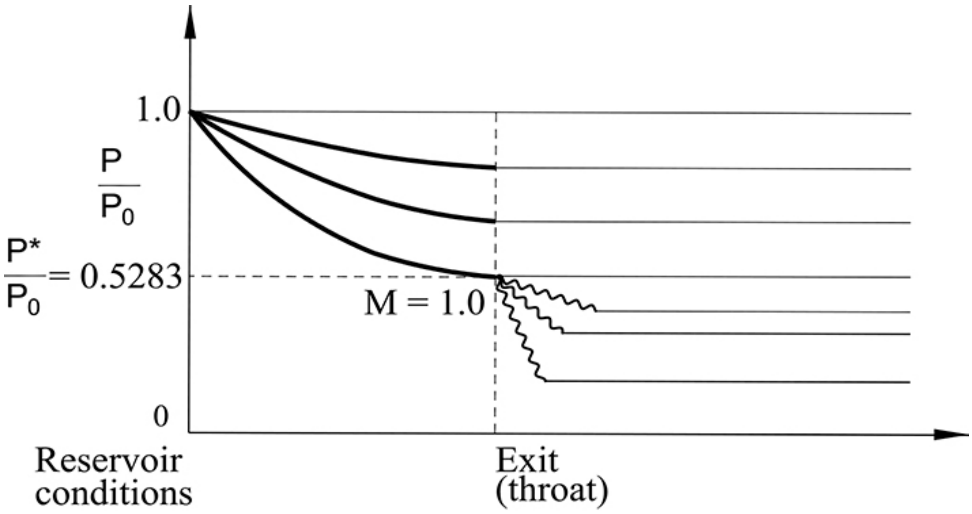


Fig. 13.2. Pressure in converging nozzle

If P_b/P_0 < 0.5283 (for gasses with k = 1.4) no more mass flow passes through the nozzle. It's the maximum mass flow that the nozzle can pass. It's called ***chocked flow***.

The conservation of mass says

$$\dot{m} = \rho \cdot A \cdot V = \rho_1 \cdot A_1 \cdot V_1 = \rho_t \cdot A_t \cdot V_t, \tag{13.1}$$

where subindex *1* stands for any cross-section area in the nozzle, subindex t stands for throat area.

If the Mack Number reaches M = 1.0 then we have maximum flowrate

$$\dot{m}_{max} = \rho^* \cdot A^* \cdot c^*. \tag{13.2}$$

where * means maximum flow, c* is speed of sound (it's because the flow is sonic now).

Using Equation (12.23) we can write

$$\dot{m}_{max} = \rho^* \cdot A^* \cdot \sqrt{k \cdot R \cdot T^*}. \tag{13.3}$$

Then, taking into account Equations (12.41a), (12.41b) and (12.41c) we obtain the equation for maximum mass flow (k = 1.4)

$$\dot{m}_{max} = (0.6847)\frac{P_0 \cdot A^*}{(R \cdot T_0)^{1/2}}, \tag{13.4}$$

where A^* is the exit (throat) area.

We can plot ratio P_b/P_0 on the horizontal axis and mass flow $\dot{m}$ on vertical axis (Fig. 13.3).

If $P_b = P_0$ ($P_b/P_0 = 1$) we have no flow. When we start reducing the back pressure ($P_b/P_0 < 1$) we get some flow. The mass flow increases until we reach the Mach Number M = 1.0. P^* is the back pressure where the Mach Number is 1.0. Beyond that point no more mass flow can be passed through the nozzle if the back pressure is reduced further. The curve flattens. The flat part is called chocked flow.

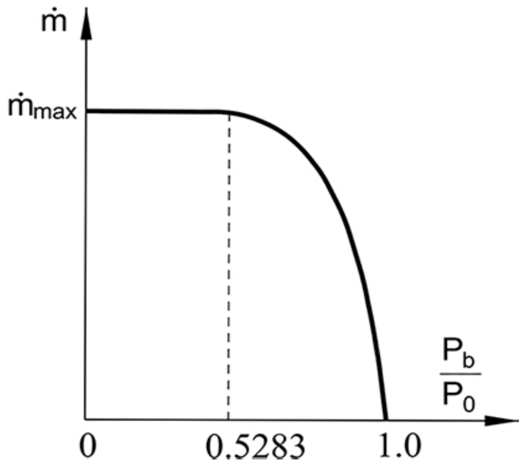


Fig. 13.3. Dependance of mass flow on pressure drop

Shock waves

Shock waves are irreversible discontinuities which can occur in a supersonic flow field, either internal or external.

Shock waves can be stationary or moving. Their thickness is on the order of 1×10^{-5} cm. Across this distance dramatic things happen. There are big changes in properties of fluid. There is tremendous deceleration of molecules across the shock. We can analyze the shock waves from the basic equations of fluid mechanics.

Consider a shock wave in a constant area duct (Fig. 13.4).

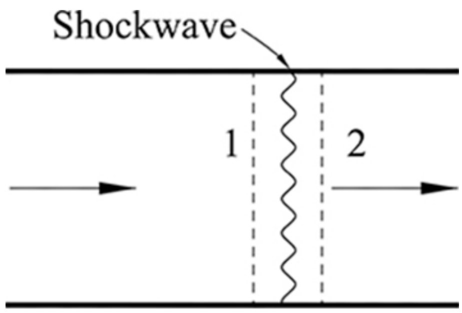


Fig. 13.4. Shock wave in a constant area duct

We have upstream conditions (1) and downstream conditions (2). Assuming that it's a steady flow, we can write *conservation of mas* as follows

$$\rho_1 \cdot A_1 \cdot V_1 = \rho_2 \cdot A_2 \cdot V_2$$

If it's a constant area duct we have

$$A_1 = A_2,$$

so

$$\rho_1 \cdot V_1 = \rho_2 \cdot V_2.$$

Momentum equation in *x* direction

$$P_1 \cdot A_1 - P_1 \cdot A_1 = \dot{m} \cdot V_2 - \dot{m} \cdot V_1.$$

The first law of thermal (energy) says

$$h_1 \cdot \frac{V_1}{2} = h_2 \cdot \frac{V_2}{2},$$

where h_1, h_2 are enthalpy upstream and downstream respectively.

Equation of state for a perfect gas

$$P = \rho \cdot R \cdot T.$$

Change in enthalpy (by definition of C_P)

$$h_2 - h_1 = C_P(T_2 - T_1).$$

There are five equations: conservation of mass, momentum, energy, equation of state, constant C_p. Assuming that conditions (1) are known, we also have five unknowns in those equations (downstream unknowns), ρ_2, P_2, T_2, V_2, and h_2. So, the object is to solve those 5 equations for those 5 unknowns.

Let's express right-hand side of the equations in terms of Mach number (they all for gases with k = 1.4)

$$\frac{P_2}{P_1} = \frac{7M_1^2 - 1}{6},$$

$$\frac{T_2}{T_1} = \frac{(M_1^2 + 5)(7M_1^2 - 1)}{36M_1^2},$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{2.4M_1^2}{2 + 0.4M_1^2},$$

$$M_2^2 = \frac{M_1^2 + 5}{7M_1^2 - 1}.$$

For gases with k ≠ 1.4 these equations are different

$$\frac{P_2}{P_1} = \frac{1}{k+1}[2k \cdot M_1^2 - (k-1)],$$

$$\frac{T_2}{T_1} = [2 + (k-1)M_1^2]\frac{2k \cdot M_1^2 - (k-1)}{(k+1)^2 M_1^2},$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(k+1)M_1^2}{(k-1)M_1^2 + 2},$$

$$\frac{A_2^*}{A_1^*} = \frac{M_2}{M_1}\left[\frac{2 + (k-1)M_1^2}{2 + (k-1)M_2^2}\right]^{(1/2)^{(k+1)}/(k-1)}.$$

These equations describe changes of the characteristics across a shock wave that is listed in Table 13.1.

Table 13.1. Changes across a shock wave

Variable	**Change across a shock wave**
Mach number	Decreases
Static pressure	Increases
Stagnation pressure	Decreases
Static temperature	Increases
Stagnation temperature	Does not change
Density	Increases
Velocity	Decreases
Entropy	Increases

When we solve these 5 equations for 5 unknowns, we are going to get two values of V because it's squared. If we put those in the change in entropy one gives a positive change and one a negative. We can guess that entropy should go up because the shock wave is irreversible. So, we have to choose the right V.

Shock wave in nozzles

There *cannot be a shock wave in a converging nozzle*. It's because we can never reach supersonic speed in such a nozzle.

Consider a converging diverging nozzle (Fig. 13.5). We have large reservoir, throat and exit area. When there is no flow the ratio $P/P_0 = 1$ and stay the same along the nozzle. If we lower the back pressure the ratio P/P_0 (velocity increases) along the nozzle up to the throat. Then the ratio recovers partly (velocity decreases) from the nozzle to the exit plane. So, the Mach Number goes up first and then goes down again. It happens as long as the speed remains subsonic.

By lowering the back pressure P_b we can reach sonic speed (M = 1.0) at the throat. At that moment the pressure ratio reaches $P/P_0 = 0.528$. If the back pressure is not low enough, the ratio goes up after the throat and the flow speed recovers to subsonic.

By lowering the back pressure P_b even further we can get supersonic flow in diverging part of the nozzle.

If the pressure ratio at exit area is higher then it's required for having supersonic speed at the exit plane, we will have a shock wave in diverging part of the nozzle. The shock wave recovers the pressure from the basic pressure curve (the lowest curve in Fig. 13.5). The more ii the velocity (the more is the Mach Number) the stronger is the shock wave and the more it moves to the exit plane.

If the shock wave occurs in converging part of the nozzle the exit pressure is equal to the back pressure. If the shock wave sits on the exit plane the exit pressure is not equal to the back pressure.

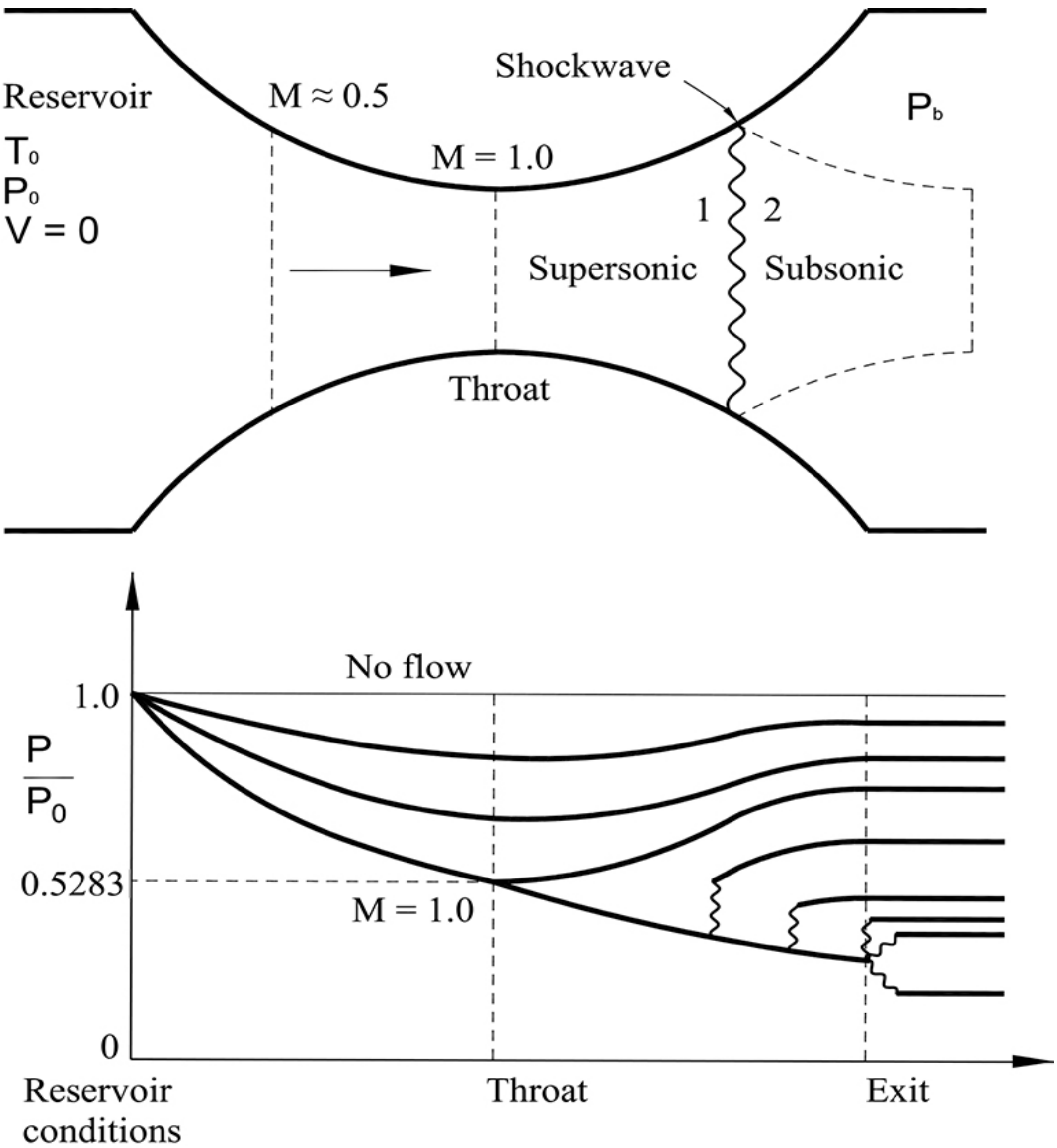


Fig. 13.5. Shock wave in a converging diverging nozzle

THEME 14. EXTERNAL FLOW OVER A FLAT PLATE (THE BOUNDARY LAYER)

Laminar flow over a flat plate

Consider a flat plate (Fig. 14.1). The plate has length *L* and width ω (into the paper). The plate is assumed to be sharp-edged, so, there is a nice clean cut on the free stream velocity (U) approaching it. Free stream meant a stream that is far away from the plate itself.

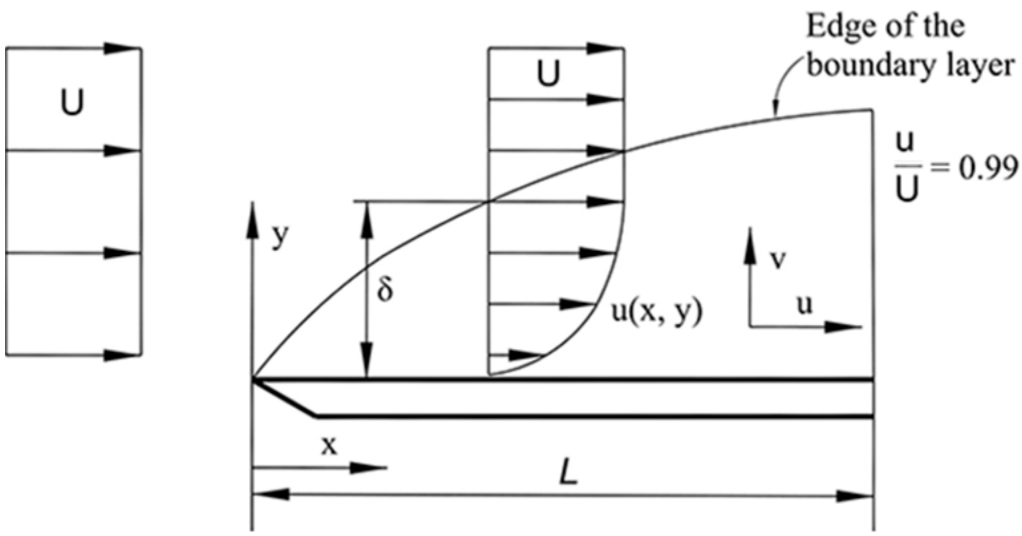


Fig. 14.1. Laminar flow over a flat plate

When the flow strikes the plate, a boundary layer builds up. It's a layer near the boundary of a solid body in external flow.

The boundary layer is critically important in analyzing the drag force on an object. Because that's where the effects of viscosity are predominant. Outside the boundary layer viscosity is not important.

Flow regime in a boundary layer can be defined by the Reynolds Number over a linear object

$$Re_L = \frac{U \cdot L}{\nu},$$

where U is the velocity outside the boundary layer (free stream velocity).

If $Re_L \leq 5 \times 10^5$ at the end of the plate, then the boundary layer is laminar.

The velocity of the fluid at the plate surface is non-slip condition. It means if the plate is stationary, the velocity of the fluid molecules adjacent to the plate is zero. The velocity anywhere in the boundary layer is given by u and it is a function of x and y coordinates.

Developing the equations for a boundary layer

The first is Navier-Stokes equation for steady incompressible flow

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}\right). \tag{14.1}$$

or

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+\nu\frac{\partial^2 u}{\partial x^2}-\frac{1}{\rho}\cdot\frac{\partial P}{\partial x}. \tag{14.2}$$

where, $\partial P/\partial x$, is assumed to be zero ($\partial P/\partial x=0$), $\partial^2 u/\partial y^2$, is negligible because $\partial^2 u/\partial x^2 << \partial^2 u/\partial y^2$.

If we neglect these things we obtain x-momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial x^2} \tag{14.3}$$

where, $u\frac{\partial u}{\partial x}$, net momentum flux in x direction, $v\frac{\partial u}{\partial y}$, net momentum flux in y direction, $\nu\frac{\partial^2 u}{\partial x^2}$, viscous effects.

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{14.4}$$

Boundary conditions are

at $y = 0$ $\qquad u = v = 0$
at $y = \delta$ $\qquad u = U$

where δ is thickness of boundary layer.

If we solve these equations by numerical integration, we will get the results that are presented in tabular form (Table 14.1).

If we are given coordinates x and y in boundary layer and we know free stream velocity U and fluid characteristic v we can find local velocity u from the Table 14.1.

Table 14.1. Blasius Velocity Profile

$y\left(\frac{U}{\nu_x}\right)^{\frac{1}{2}}$	$\frac{u}{U}$	$y\left(\frac{U}{\nu_x}\right)^{\frac{1}{2}}$	$\frac{u}{U}$
0.0	0.0	2.8	0.81152
0.2	0.06641	3.0	0.84605
0.4	0.13277	3.2	0.87609
0.6	0.19894	3.4	0.90177
0.8	0.26471	3.6	0.92333
1.0	0.32979	3.8	0.94112
1.2	0.39378	4.0	0.95552
1.4	0.45627	4.2	0.96696
1.6	0.51676	4.4	0.97587
1.8	0.57477	4.6	0.98269
2.0	0.62977	4.8	0.98779
2.2	0.68132	5.0	0.99155
2.4	0.72899	∞	1.00000
2.6	0.77246		

Another definition of boundary layer is where $u/U = 0.99$.

It gives us an important equation for a boundary layer thickness

$$\delta = \frac{5}{\sqrt{\frac{U}{\nu_x}}} = \frac{5x}{Re_x^{1/2}}, \tag{14.5}$$

where $Re_x = U_x/\nu$.

Surface shear stress at the wall (the wall surface is at $y = 0$) can be written as

$$\tau_S = \mu \frac{\partial u}{\partial y}\bigg]_{y=0}. \tag{14.6}$$

We determined $\partial u/\partial y$ from Table 7.1 and get the result

$$\tau_S = \mu \frac{\partial u}{\partial y}\bigg]_{y=0} = 0.332\sqrt{\frac{\rho \cdot \mu}{x}} \cdot U^{\frac{1}{2}}. \tag{14.7}$$

But we defined the shear stress using skin friction coefficient C_f as (came from the definition of C_f)

$$\tau_S = C_f \cdot \rho \cdot \frac{U^2}{2}. \tag{14.8}$$

Using Equations (14.6), (14.7) and (14.8) we can write

$$C_f = \frac{0.664}{Re_x^{1/2}}. \tag{14.9}$$

If we plot the shear stress τ_S versus x we get.

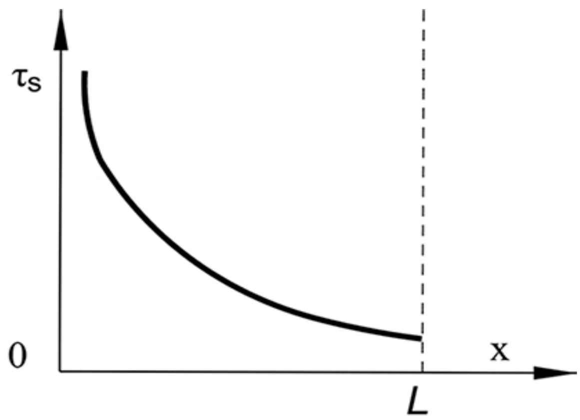


Fig. 14.2. Dependence of shear stress on x coordinate

If x goes to zero the shear stress becomes very large. As x gets bigger the shear stress goes down nonlinearly. So, the biggest shear stress on the plate is right up at the leading edge (the location where $x = 0$). As we go left the shear stress gets smaller. It's because of the slope of the u versus y graph. The slope is very steep at the leading edge and gets smaller to the end.

To get the drag force on the whole plate we have to integrate surface shear stress times the area (since $dA = b \cdot dx$)

$$D_x = b\int_0^x \tau_S(x)dx. \tag{14.10}$$

where b is the width of the plate (into the paper).

If we put Equation (14.7) into Equation (14.10), we get

$$D_x = 0.664 \cdot b\sqrt{\rho \cdot \mu} \cdot U^{\frac{3}{2}}. \tag{14.11}$$

We defined the drag force D_L over the whole length of the plate

$$D_L = C_D \cdot b \cdot L \cdot \rho\frac{U^2}{2}, \tag{14.12}$$

where, $b \cdot L$, is the area of the plate, C_D, drag coefficient.

If we equate Equations (14.11) and (14.12), we obtain

$$C_D = \frac{1.328}{Re_L^{1/2}}. \tag{14.13}$$

Turbulent flow over a flat plate

Assume turbulent flow from the leading edge. For this case mathematical analysis is very different. We don't do it totally mathematically like we did it for laminar flow. There is no exact solution. Numerical methods and/or integral approach is used.

We use 1/7 power law velocity profile with the integral approach. That profile for turbulent flow can be written in equation as

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}. \tag{14.14}$$

Integral approach gives

$$\delta = \frac{0.16x}{Re_x^{1/7}}, \tag{14.15}$$

and

$$\tau_{wall} = 0.0135 \cdot \left[\frac{\mu \cdot \rho^6 \cdot U^{13}}{x}\right]^{\frac{1}{7}}, \tag{14.16}$$

and

$$C_f = \frac{0.027}{Re_x^{1/7}}, \tag{14.17}$$

and

$$F_D = C_D \cdot A \cdot \rho \frac{U^2}{2}, \tag{14.18}$$

where

$$C_D = \frac{0.031}{Re_L^{1/7}}. \tag{14.19}$$

Mixed flow over a flat plate

Consider mixed flow (Fig. 14.3). It has partly laminar and partly turbulent boundary layer. There x_C (subindex c stands for critical) is the location where the boundary layer transits from laminar to turbulent.

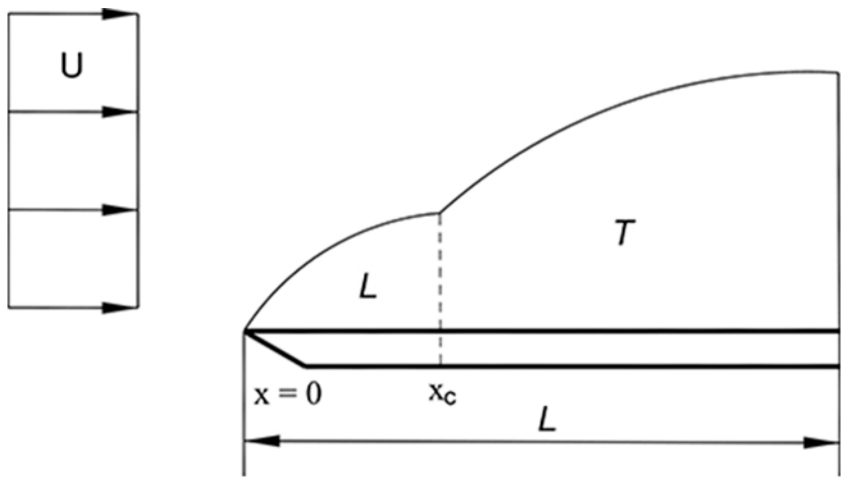


Fig. 14.3. Mixed flow over a flat plate

It means that

$$Re_x = \frac{U \cdot x_c}{\nu} = 5 \times 10^5.$$

If we solve this for x_c we get

$$x_c = \frac{5 \times 10^5 \cdot \nu}{U}. \tag{14.20}$$

For this case then

$$F_D = C_D \cdot b \cdot L \cdot \rho \frac{U^2}{2}, \tag{14.21}$$

where

$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L}. \tag{14.22}$$

Because it's part laminar part turbulent we have to integrate in pieces (separately laminar part and separately turbulent part).

So, there are three cases to worry about.

- all flow is laminar from the leading edge to the end of the plate,
- the flow starts laminar and then transits to turbulent,
- the flow is turbulent from the start.

There is also u rule: if the laminar part of the flow is less than or equal to 10% of plate length ($x_c \leq 10\%L$) assume the boundary layer all turbulent.

Fig. 14.4 shows dependence of the drag coefficient of laminar and turbulent boundary layers on the Reynolds Number for smooth and rough flat plates.

For laminar part of the curves (up to $Re_L = 5 \times 10^5$ only) Equation (14.13) is used. For turbulent smooth part of the curves Equation (14.19) is used. For transition part of the curves Equation (14.22) is used.

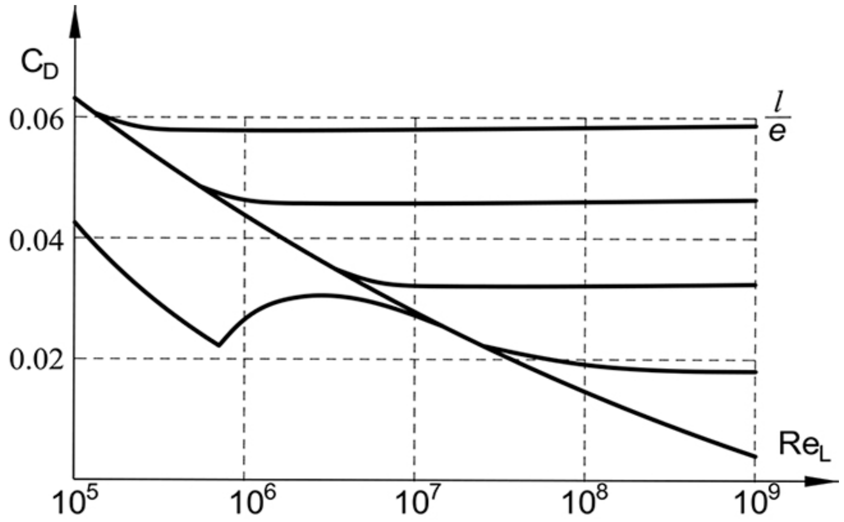


Fig. 14.4. Dependence of C_D on Re_L

THEME 15. FLAT PLATE NORMAL TO FLOW

If a plate is normal to the flow field, the drag force is created by the pressure, not by the skin friction. In front of the plate, we have the high-pressure region and in back of it we have low pressure region. The difference in the pressure creates the drag force.

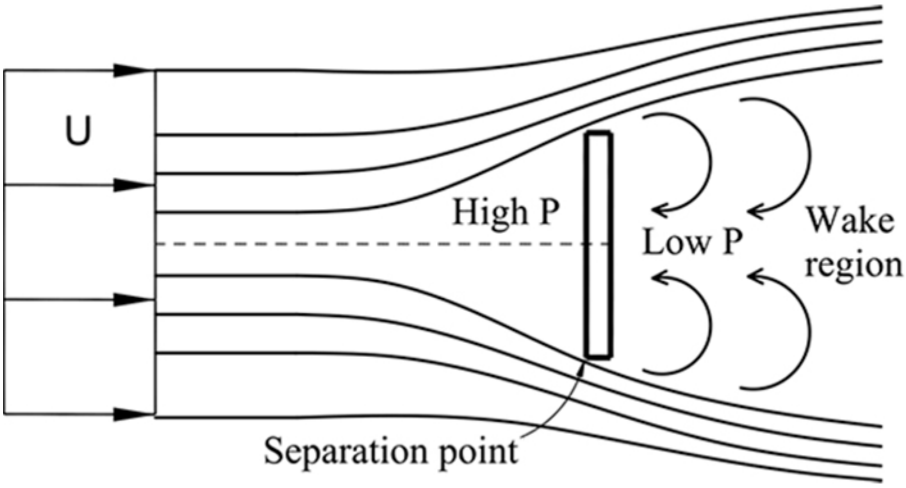


Fig. 15.1. Flow over normal flat plate

At separation point the flow separates physically from the plate. Behind that, there is a region of swirling addies. It’s called wake region where intensive mixing takes place.

So, when the plate is normal to the flow, we have pressure drag. If the plate is tangential to the flow, we have friction drag. In the most real situation, we have combination of both. Some characteristics of different types of drag are presented in Table 15.1.

Table 15.1. Characteristics of different types of drag

Drag characteristics	Friction drag	Pressure drag
Drag coefficient C_D	small	large
Dependence on Re_D	large	small
Effect of the plate roughness	large	small

Flow over circular tubes

Consider a circular tube and a flow with different Reynolds Number.

Case A. The Reynolds Number is in the order of 1.0 (velocity is very low). We have laminar flow all the way around the cylinder. The laminar boundary layer does not separate from the surface.

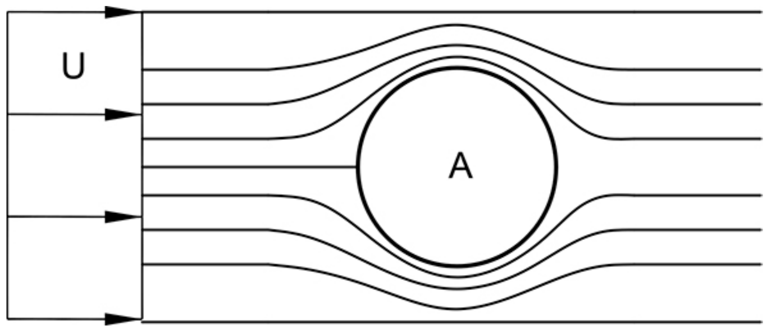


Fig. 15.2. Laminar flow over a cylinder

Case B. The Reynolds Number is 20 < Re < 50 000. At a certain point laminar boundary layer separates from the surface of the cylinder (separation point). Behind it there is a wake region with turbulent eddies.

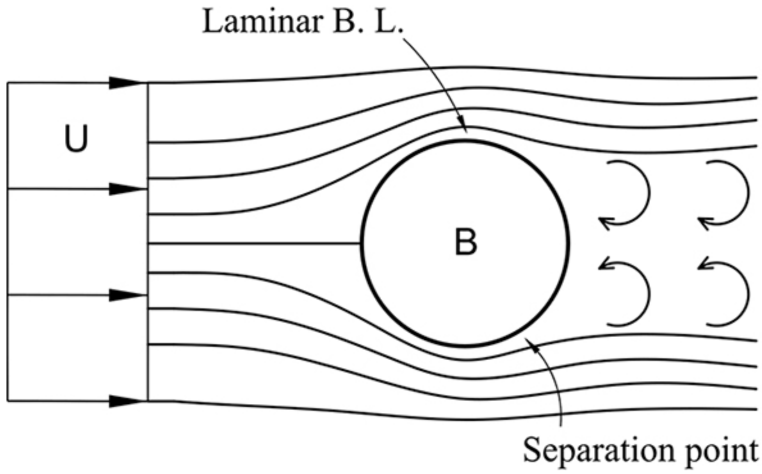


Fig. 15.3. Laminar flow over a cylinder with wake region

Case C. The Reynolds Number is 50 000 < Re < 100 000. The separation point moves to the front of the cylinder when the Reynolds Number increases. The wake region gets bigger.

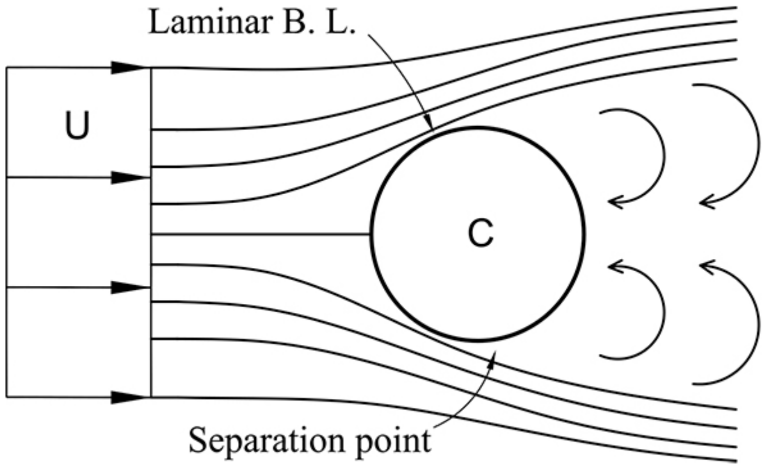


Fig. 15.4. Flow over a cylinder with large wake region

Case D. The Reynolds Number is Re > 100 000. The Reynolds Number is so large that the boundary layer, which was laminar, transits tot turbulent. Turbulent boundary layer sticks to the wall of the cylinder better. So, the wake region behind the cylinder become smaller.

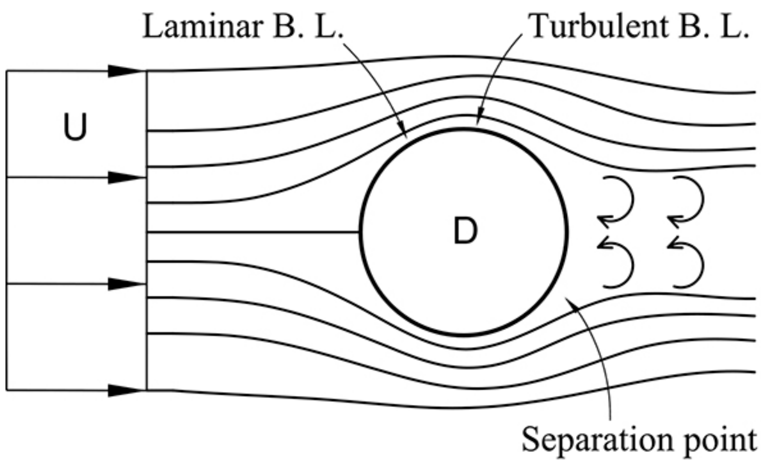


Fig. 15.5. Flow over a cylinder with reduced wake region

The main difference between profiles is that in turbulent profile the velocity is bigger near the surface than in laminar (Fig. 15.6). The bigger velocity means more momentum. So, the flow sticks to the surface longer because of that momentum.

If we want to find a drag force, we can use Equation (14.18) (kinetic energy times the area times the drag coefficient). The area there is a frontal area of stubby bodies such as spheres, cylinders, cars, trucks etc.

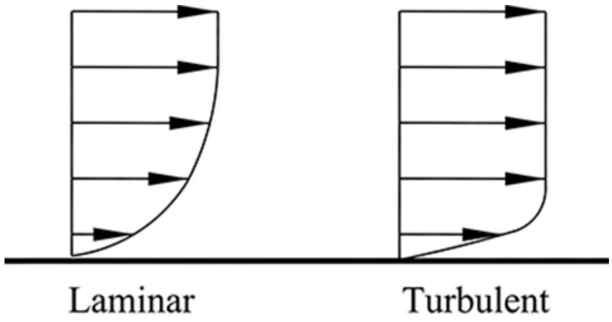


Fig. 15.6. Typical velocity profiles

Dependence of drag coefficient on the Reynolds Number for a smooth cylinder and a smooth sphere is shown in the Fig. 15.7. If we correlate the curves with cases A, B, C and D (Fig. 15.2, 15.3, 15.4 and 15.5) we can label them as regions A, B, C and D on the graph.

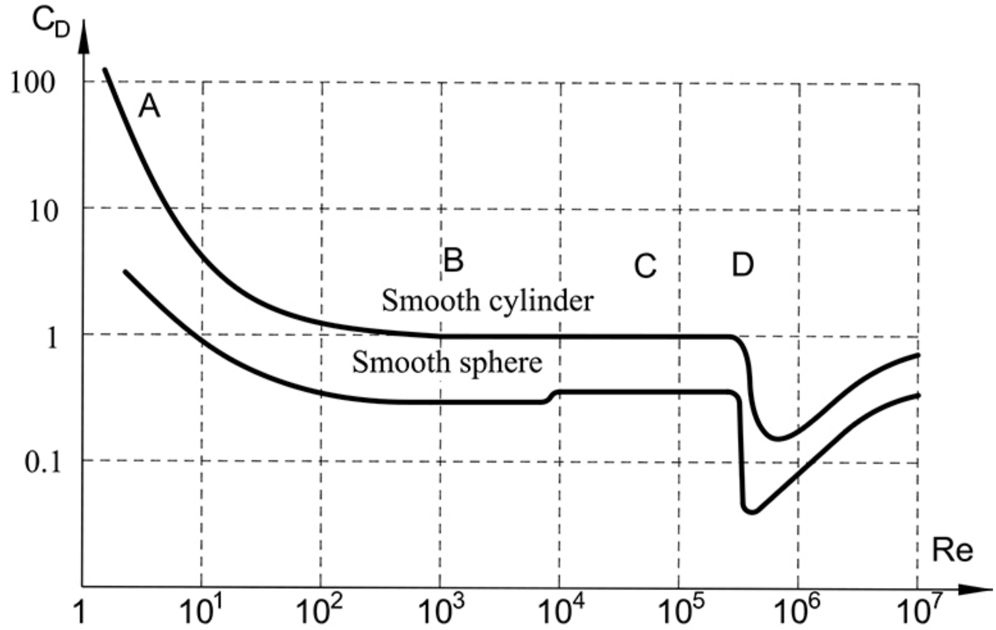


Fig. 15.7. Dependence of the drag coefficient on the Reynolds Number

The highest drag coefficient is when the Reynolds Number is very low and the boundary layer is all laminar (region A). It continues to lower with increasing Re and levels relatively at Re ≈ 10^3 when laminar boundary layer separates from the surface and a wake region appears (region B). It increases slightly when the wake region gets larger (region C). It drops dramatically when the laminar boundary layer transits tot turbulent the wake region become smaller (region D).

We can see that in region D there is 70% drop in the drag coefficient. The main reason is the wake region. Laminar boundary layer suddenly transited to turbulent and kept being attached to the surface all the way to the back side. It cats the wake region down from a humongous area to a very small area.

The wake region causes a lot of drag.

Let's take the region from Re = 10^2 to Re = 10^6 for smooth sphere and blow int up (Fig. 15.8).

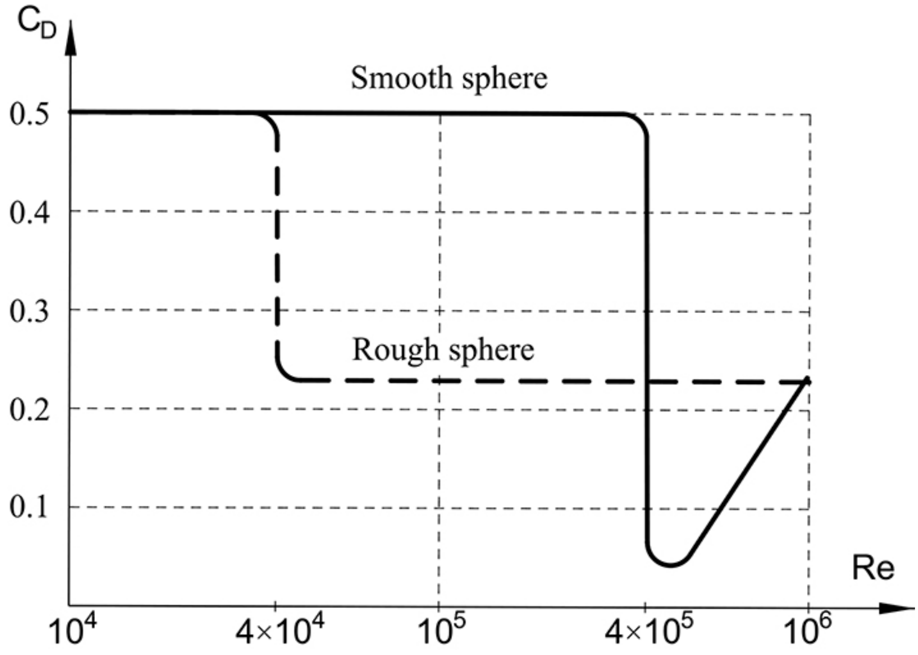


Fig. 15.8. Part of the dependence of the drag coefficient on the Reynolds number

If the sphere gets rougher, it's roughness creates turbulence in the boundary layer. It causes boundary layer to stick longer to the surface. The sudden drop in drag coefficient occurs earlier, but it is not so dramatic compared with the smooth surface. The curve change to the dashed line.

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APPENDIX A
Compressible-flow tables

Table B.1 Isentropic flow of a perfect gas, k = 1.4

M	P/P_0	T/T_0	ρ/ρ_0	A/A^*
1	2	3	4	5
0.0	1.0	1.0	1.0	∞
0.02	0.9997	0.9998	0.9999	28.9421
0.04	0.9989	0.9992	0.9997	14.4815
0.06	0.9975	0.9982	0.9993	9.6659
0.08	0.9955	0.9968	0.9987	7.2616
0.1	0.9930	0.9950	0.9980	5.8218
0.12	0.9900	0.9928	0.9971	4.8643
0.14	0.9864	0.9903	0.9961	4.1824
0.16	0.9823	0.9873	0.9949	3.6727
0.18	0.9776	0.9840	0.9936	3.2779
0.2	0.9725	0.9803	0.9921	2.9635
0.22	0.9668	0.9762	0.9904	2.7076
0.24	0.9607	0.9718	0.9886	2.4956
0.26	0.9541	0.9670	0.9867	2.3173
0.28	0.9470	0.9619	0.9846	2.1656
0.3	0.9395	0.9564	0.9823	2.0351
0.32	0.9315	0.9506	0.9799	1.9219
0.34	0.9231	0.9445	0.9774	1.8229
0.36	0.9143	0.9380	0.9747	1.7358
0.38	0.9052	0.9313	0.9719	1.6587
0.4	0.8956	0.9243	0.9690	1.5901
0.42	0.8857	0.9170	0.9659	1.5289
0.44	0.8755	0.9094	0.9627	1.4740
0.46	0.8650	0.9016	9594	1.4246
0.48	0.8541	0.8935	0.9559	1.3801
0.5	0.8430	0.8852	0.9524	1.3398
0.52	0.8317	0.8766	0.9487	1.3034
0.54	0.8201	0.8679	0.9449	1.2703
0.56	0.8082	0.8589	0.9410	1.2403
0.58	0.7962	0.8498	0.9370	1.2130
0.6	0.7840	0.8405	0.9328	1.1882
0.62	0.7716	0.8310	0.9286	1.1656
0.64	0.7591	0.8213	0.9243	1.1451
0.66	0.7465	0.8115	0.9199	1.1265
0.68	0.7338	0.8016	0.9153	1.1097

Continuation of the Table B.1

1	2	3	4	5
0.7	0.7209	0.7916	0.9107	1.0944
0.72	0.7080	0.7814	0.9061	1.0806
0.74	0.6951	0.7712	0.9013	1.0681
0.76	0.6821	0.7609	0.8964	1.0570
0.78	0.6690	0.7505	0.8915	1.0471
0.8	0.6560	0.7400	0.8865	1.0382
0.82	0.6430	0.7295	0.8815	1.0305
0.84	0.6300	0.7189	0.8763	1.0237
0.86	0.6170	0.7083	0.8711	1.0179
0.88	0.6041	0.6977	0.8659	1.0129
0.9	0.5913	0.6870	0.8606	1.0089
0.92	0.5785	0.6764	0.8552	1.0056
0.94	0.5658	0.6658	0.8498	1.0031
0.96	0.5532	0.6551	0.8444	1.0014
0.98	0.5407	0.6445	0.8389	1.0003
1.0	0.5283	0.6339	0.8333	1.0000
1.02	0.5160	0.6234	0.8278	1.003
1.04	0.5039	0.6129	0.8222	1.0013
1.06	0.4919	0.6024	0.8165	1.0029
1.08	0.4800	0.5920	0.8108	1.0051
1.1	0.4684	0.5817	0.8052	1.0079
1.12	0.4568	0.5714	0.7994	1.0113
1.14	0.4455	0.5612	0.7937	1.0153
1.16	0.4343	0.5511	0.7879	1.0198
1.18	0.4232	0.5411	0.7822	1.0248
1.2	0.4124	0.5311	0.7764	1.0304
1.22	0.4017	0.5213	0.7706	1.0366
1.24	0.3912	0.5115	0.7648	1.0432
1.26	0.3809	0.5019	0.7590	1.0504
1.28	0.3708	0.4923	0.7532	1.0581
1.3	0.3609	0.4829	0.7474	1.0663
1.32	0.3512	0.4736	0.7416	1.0750
1.34	0.3417	0.4644	0.7358	1.0842
1.36	0.3323	0.4553	0.7300	1.0940
1.38	0.3232	0.4463	0.7242	1.1042
1.4	0.3142	0.4374	0.7184	1.1149
1.42	0.3055	0.4287	0.7126	1.1262
1.44	0.2969	0.4201	0.7069	1.1379
1.46	0.2886	0.4116	0.7011	1.1501
1.48	0.2804	0.4032	0.6954	1.1629

Continuation of the Table B.1

1	2	3	4	5
1.5	0.2724	0.3950	0.6897	1.1762
1.52	0.2646	0.3869	0.6840	1.1899
1.54	0.2570	0.3789	0.6783	1.2042
1.56	0.2496	0.3710	0.6726	1.2190
1.58	0.2423	0.3633	0.6670	1.2344
1.6	0.2353	0.3557	0.6614	1.2502
1.62	0.2284	0.3483	0.6558	1.2666
1.64	0.2217	0.3409	0.6502	1.2836
1.66	0.2151	0.3337	0.6447	1.3010
1.68	0.2088	0.3266	0.6392	1.3190
1.7	0.2026	0.3197	0.6337	1.3376
1.72	0.1966	0.3129	0.6283	1.3567
1.74	0.1907	0.3062	0.6229	1.3764
1.76	0.1850	0.2996	0.6175	1.3967
1.78	0.1794	0.2931	0.6121	1.4175
1.8	0.1740	0.2868	0.6068	1.4390
1.82	0.1688	0.2806	0.6015	1.4610
1.84	0.1637	0.2745	0.5963	1.4836
1.86	0.1587	0.2686	0.5910	1.5069
1.88	0.1539	0.2627	0.5859	1.5308
1.9	0.1492	0.2570	0.5807	1.5553
1.92	0.1447	0.2514	0.5756	1.5804
1.94	0.1403	0.2459	0.5705	1.6062
1.96	0.1360	0.2405	0.5655	1.6326
1.98	0.1318	0.2352	0.5605	1.6597
2.0	0.1278	0.2300	0.5556	1.6875
2.02	0.1239	0.2250	0.5506	1.7160
2.04	0.1201	0.2200	0.5458	1.7451
2.06	0.1164	0.2152	0.5409	1.7750
2.08	0.1128	0.2104	0.5361	1.8056
2.1	0.1094	0.2058	0.5313	1.8369
2.12	0.1060	0.2013	0.5266	1.8690
2.14	0.1027	0.1968	0.5219	1.9018
2.16	0.0996	0.1925	0.5173	1.9354
2.18	0.0965	0.1882	0.5127	1.9698
2.2	0.0935	0.1841	0.5081	2.0050
2.22	0.0906	0.1800	0.5036	2.0409
2.24	0.0878	0.1760	0.4991	2.0777
2.26	0.0851	0.1721	0.4947	2.1153
2.28	0.0825	0.1683	0.4903	2.1538

Continuation of the Table B.1

1	2	3	4	5
2.3	0.0800	0.1646	0.4859	2.1931
2.32	0.0775	0.1609	0.4816	2.2333
2.34	0.0751	0.1574	0.4773	2.2744
2.36	0.0728	0.1539	0.4731	2.3164
2.38	0.0706	0.1505	0.4688	2.3593
2.4	0.0684	0.1472	0.4647	2.4031
2.42	0.0663	0.1439	0.4606	2.4479
2.44	0.0643	0.1408	0.4565	2.4936
2.46	0.0623	0.1377	0.4524	2.5403
2.48	0.0604	0.1346	0.4484	2.5880
2.5	0.0585	0.1317	0.4444	2.6367
2.52	0.0567	0.1288	0.4405	2.6865
2.54	0.0550	0.1260	0.4366	2.7372
2.56	0.0533	0.1232	0.4328	2.7891
2.58	0.0517	0.1205	0.4289	2.8420
2.6	0.0501	0.1179	0.4252	2.8960
2.62	0.0486	0.1153	0.4214	2.9511
2.64	0.0471	0.1128	0.4177	3.0073
2.66	0.0457	0.1103	0.4141	3.0647
2.68	0.0443	0.1079	0.4104	3.1233
2.7	0.0430	0.1056	0.4068	3.1830
2.72	0.0417	0.1033	0.4033	3.2440
2.74	0.0404	0.1010	0.3998	3.3061
2.76	0.0392	0.0989	0.3963	3.3695
2.78	0.0380	0.0967	0.3928	3.4342
2.8	0.0368	0.0946	0.3894	3.5001
2.82	0.0357	0.0926	0.3860	3.5674
2.84	0.0347	0.0906	0.3827	3.6359
2.86	0.0336	0.0886	0.3794	3.7058
2.88	0.0326	0.0867	0.3761	3.7771
2.9	0.0317	0.0849	0.3729	3.8498
2.92	0.0307	0.0831	0.3696	3.9238
2.94	0.0298	0.0813	0.3665	3.9993
2.96	0.0289	0.0796	0.3633	4.0763
2.98	0.0281	0.0779	0.3602	4.1547
3.0	0.0272	0.0762	0.3571	4.2346
3.02	0.0264	0.0746	0.3541	4.3160
3.04	0.0256	0.0730	0.3511	4.3990
3.06	0.0249	0.0715	0.3481	4.4835
3.08	0.0242	0.0700	0.3452	4.5696

Continuation of the Table B.1

1	2	3	4	5
3.1	0.0234	0.0685	0.3422	4.6573
3.12	0.0228	0.0671	0.3393	4.7467
3.14	0.0221	0.0657	0.3365	4.8377
3.16	0.0215	0.0643	0.3337	4.9304
3.18	0.0208	0.0630	0.3309	5.0248
3.2	0.0202	0.0617	0.3281	5.1210
3.22	0.0196	0.0604	0.3253	5.2189
3.24	0.0191	0.0591	0.3226	5.3186
3.26	0.0185	0.0579	0.3199	5.4201
3.28	0.0180	0.0567	0.3173	5.5234
3.3	0.0175	0.0555	0.3147	5.6286
3.32	0.0170	0.0544	0.3121	5.7358
3.34	0.0165	0.0533	0.3095	5.8448
3.36	0.0160	0.0522	0.3069	5.9558
3.38	0.0156	0.0511	0.3044	6.0687
3.4	0.0151	0.0501	0.3019	6.1837
3.42	0.0147	0.0491	0.2995	6.3007
3.44	0.0143	0.0481	0.2970	6.4198
3.46	0.0139	0.0471	0.2946	6.5409
3.48	0.0135	0.0462	0.2922	6.6642
3.5	0.0131	0.0452	0.2899	6.7896
3.52	0.0127	0.0443	0.2875	6.9172
3.54	0.0124	0.0434	0.2852	7.0471
3.56	0.0120	0.0426	0.2829	7.1791
3.58	0.0117	0.0417	0.2806	7.3135
3.6	0.0114	0.0409	0.2784	7.4501
3.62	0.0111	0.0401	0.2762	7.5891
3.64	0.0108	0.0393	0.2740	7.7305
3.66	0.0105	0.0385	0.2718	7.8742
3.68	0.0102	0.0378	0.2697	8.0204
3.7	0.0099	0.0370	0.2675	8.1691
3.72	0.0096	0.0363	0.2654	8.3202
3.74	0.0094	0.0356	0.2633	8.4739
3.76	0.0091	0.0349	0.2613	8.6302
3.78	0.0089	0.0342	0.2592	8.7891
3.8	0.0086	0.0335	0.2572	8.9506
3.82	0.0084	0.0329	0.2552	9.1148
3.84	0.0082	0.0323	0.2532	9.2817
3.86	0.0080	0.0316	0.2513	9.4513
3.88	0.0077	0.0310	0.2493	9.6237

Continuation of the Table B.1

1	2	3	4	5
3.9	0.0075	0.0304	0.2474	9.7990
3.92	0.0073	0.0299	0.2455	9.9771
3.94	0.0071	0.0293	0.2436	10.1581
3.96	0.0096	0.0287	0.2418	10.3420
3.98	0.0068	0.0282	0.2399	10.5289
4.0	0.0066	0.0277	0.2381	10.7188
4.02	0.0064	0.0271	0.2363	10.9117
4.04	0.0062	0.0266	0.2345	11.1077
4.06	0.0061	0.0261	0.2327	11.3068
4.08	0.0059	0.0256	0.2310	11.5091
4.1	0.0058	0.0252	0.2293	11.7147
4.12	0.0056	0.0247	0.2275	11.9234
4.14	0.0055	0.0242	0.2258	12.1354
4.16	0.0053	0.0238	0.2242	12.3508
4.18	0.0052	0.0234	0.2225	12.5695
4.2	0.0051	0.0229	0.2208	12.7916
4.22	0.0049	0.0225	0.2192	13.0172
4.24	0.0048	0.0221	0.2176	13.2463
4.26	0.0047	0.0217	0.2160	13.4789
4.28	0.0046	0.0213	0.2144	13.7151
4.3	0.0044	0.0209	0.2129	13.9549
4.32	0.0043	0.0205	0.2113	14.1984
4.34	0.0042	0.0202	0.2098	14.4456
4.36	0.0041	0.0198	0.2083	14.6965
4.38	0.0040	0.0194	0.2067	14.9513
4.4	0.0039	0.0191	0.2053	15.2099
4.42	0.0038	0.0187	0.2038	15.4724
4.44	0.0037	0.0184	0.2023	15.7388
4.46	0.0036	0.0181	0.2009	16.0092
4.48	0.0035	0.0178	0.1994	16.2837
4.5	0.0035	0.0174	0.1980	16.5622
4.52	0.0034	0.0171	0.1966	16.8449
4.54	0.0033	0.0168	0.1952	17.1317
4.56	0.0032	0.0165	0.1938	17.4228
4.58	0.0031	0.0163	0.1925	17.7181
4.6	0.0031	0.0160	0.1911	18.0178
4.62	0.0030	0.0157	0.1898	18.3218
4.64	0.0029	0.0154	0.1885	18.6303
4.66	0.0028	0.0152	0.1872	18.9433
4.68	0.0028	0.0149	0.1859	19.2608

Continuation of the Table B.1

1	2	3	4	5
4.7	0.0027	0.0146	0.1846	19.5828
4.72	0.0026	0.0144	0.1833	19.9095
4.74	0.0026	0.0141	0.1820	20.2409
4.76	0.0025	0.0139	0.1808	20.5770
4.78	0.0025	0.0137	0.1795	20.9179
4.8	0.0024	0.0134	0.1783	21.2637
4.82	0.0023	0.0132	0.1771	21.6144
4.84	0.0023	0.0130	0.1759	21.9700
4.86	0.0022	0.0128	0.1747	22.3306
4.88	0.0022	0.0125	0.1735	22.6963
4.9	0.0021	0.0123	0.1724	23.0671
4.92	0.0021	0.0121	0.1712	23.4431
4.94	0.0020	0.0119	0.1700	23.8243
4.96	0.0020	0.0117	0.1689	24.2109
4.98	0.0019	0.0115	0.1678	24.6027
5.0	0.0019	0.0113	0.1667	25.0000

Table B.2 Normal-shock relations for a perfect gas, $k = 1.4$

Ma_{n1}	Ma_{n2}	p_2/p_1	$V_1/V_2=\rho_2/\rho_1$	T_2/T_1	p_0/p_{01}	A_2^*/A_1^*
1	2	3	4	5	6	7
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.02	0.9805	1.0471	1.0334	1.0132	1.0000	1.0000
1.04	0.9620	1.0952	1.0671	10263	0.9999	1.0001
1.06	0.9444	1.1442	1.1009	1.0393	0.9998	1.0002
1.08	0.9277	1.1941	1.1349	1.0522	0.9994	1.0006
1.1	0.9118	1.2450	1.1691	1.0649	0.9989	1.0011
1.12	0.8966	1.2968	1.2034	1.0776	0.9982	1.0018
1.14	0.8820	1.3495	1.2378	1.0903	0.9973	1.0027
1.16	0.8682	1.4032	1.2723	1.1029	0.9961	1.0040
1.18	0.8549	1.4578	1.3069	1.1154	0.9946	1.0055
1.2	0.8422	1.5133	1.3416	1.1280	0.9928	1.0073
1.22	0.8300	1.5698	1.3764	1.1405	0.9907	1.0094
1.24	0.8183	1.6272	1.4112	1.1531	0.9884	1.0118
1.26	0.8071	1.6855	1.4460	1.1657	0.9857	1.0145
1.28	0.7963	1.7448	1.4808	1.1783	0.9827	1.0176
1.3	0.7860	1.8050	1.5157	1.1909	0.9794	1.0211
1.32	0.7760	1.8661	1.5505	1.2035	0.9758	1.0249
1.34	0.7664	1.9282	1.5854	1.2162	0.9718	1.0290
1.36	0.7572	1.9912	1.6202	1.2290	0.9676	1.0335
1.38	0.7483	20551	1.6549	1.2418	0.9630	1.0384
1.4	0.7397	2.1200	1.6897	1.2547	0.9582	1.0436
1.42	0.7314	2.1858	1.7243	1.2676	0.9531	1.0492
1.44	0.7235	2.2525	1.7589	1.2807	0.9476	1.0552
1.46	0.7157	2.3202	1.7934	1.2938	0.9420	1.0616
1.48	0.7083	2.3888	1.8278	1.3069	0.9360	1.0684
1.5	0.7011	2.4583	1.8621	1.3202	0.9298	1.0755
1.52	0.6941	2.5288	1.8963	1.3336	0.9233	1.0830
1.54	0.6874	2.6002	1.9303	1.3470	0.9166	1.0910
1.56	0.6809	2.6725	1.9643	1.3606	0.9097	1.0993
1.58	0.6746	2.7458	1.9981	1.3742	0.9026	1.1080
1.6	0.6684	2.8200	2.0317	1.3880	0.8952	1.1171
1.62	0.6625	2.8951	2.0653	1.4018	0.8877	1.1266
1.64	0.6568	2.9712	2.0986	1.4158	0.8799	1.1365
1.66	0.6512	3.0482	2.1318	1.4299	0.8720	1.1468
1.68	0.6458	3.1261	2.1649	1.4440	0.8639	1.1575
1.7	0.6405	3.2050	2.1977	1.4583	0.8557	1.1686
1.72	0.6355	3.2848	2.2304	1.4727	0.8474	1.1801
1.74	0.6305	3.3655	2.2629	1.4873	0.8389	1.1921
1.76	0.6257	3.4472	2.2952	1.5019	0.8302	1.2045

Continuation of the Table B.2

1	2	3	4	5	6	7
1.78	0.6210	3.5298	2.3273	1.5167	0.8215	1.2173
1.8	0.6165	3.6133	2.3592	1.5316	0.8127	1.2305
1.82	0.6121	3.6978	2.3909	1.5466	0.8038	1.2441
1.84	0.6078	3.7832	2.4224	1.5617	0.7948	1.2582
1.86	0.6036	3.8695	2.4537	1.5770	0.7857	1.2728
1.88	0.5996	3.9568	2.4848	1.5924	0.7765	1.2877
1.9	0.5956	4.0450	2.5157	1.6079	0.7674	1.3032
1.92	0.5918	4.1341	2.5463	1.6236	0.7581	1.3191
1.94	0.5880	4.2242	2.5767	1.6394	0.7488	1.3354
1.96	0.5844	4.3152	2.6069	1.6553	0.7395	1.3522
1.98	0.5808	4.4071	2.6369	1.6713	0.7302	1.3695
2.0	0.5774	4.5000	2.6667	1.6875	0.7209	1.3872
2.02	0.5740	4.5938	2.6962	1.7038	0.7115	1.4054
2.04	0.5707	4.6885	2.7255	1.7203	0.7022	1.4241
2.06	0.5675	4.7842	2.7545	1.7369	0.6928	1.4433
2.08	0.5643	4.8808	2.7833	1.7536	0.6835	1.4630
2.1	0.5613	4.9783	2.8116	1.7705	0.6742	1.4832
2.12	0.5583	5.0768	2.8402	1.7875	0.6649	1.5039
2.14	0.5554	5.1762	2.8683	1.8046	0.6557	1.5252
2.16	0.5525	5.2765	2.8962	1.8219	0.6464	1.5469
2.18	0.5498	5.3778	2.9238	1.8393	0.6373	1.5692
2.2	0.5471	5.4800	2.9512	1.8569	0.6281	1.5920
2.22	0.5444	5.5831	2.9784	1.8746	0.6191	1.6154
2.24	0.5418	5.6872	3.0053	1.8924	0.6100	1.6393
2.26	0.5393	5.7922	3.0319	1.9104	0.6011	1.6638
2.28	0.5368	5.8981	3.0584	1.9285	0.5921	1.6888
2.3	0.5344	6.0050	3.0845	1.9468	0.5833	1.7144
2.32	0.5321	6.1128	3.1105	1.9652	0.5745	1.7406
2.34	0.5297	6.2215	3.1362	1.9838	0.5658	1.7674
2.36	0.5275	6.3312	3.1617	2.0025	0.5572	1.7948
2.38	0.5253	6.4418	3.1869	2.0213	0.5486	1.8228
2.4	0.5231	6.5533	3.2119	2.0403	0.5401	1.8514
2.42	0.5210	6.6658	3.2367	2.0595	0.5317	1.8806
2.44	0.5189	6.7792	3.2612	2.0788	0.5234	1.9105
2.46	0.5169	6.8935	3.2855	2.0982	0.5152	1.9410
2.48	0.5149	7.0088	3.3095	2.1178	0.5071	1.9721
2.5	0.5130	7.1250	3.3333	2.1375	0.4990	2.0039
2.52	0.5111	7.2421	3.3569	2.1574	0.4911	2.0364
2.54	0.5092	7.3602	3.3803	2.1774	0.4832	2.0696
2.56	0.5074	7.4792	3.4034	2.1976	0.4754	2.1035

Continuation of the Table B.2

1	2	3	4	5	6	7
2.58	0.5056	7.5991	3.4263	2.2179	0.4677	2.1381
2.6	0.5039	7.7200	3.4490	2.2383	0.4601	2.1733
2.62	0.5022	7.8418	3.4714	2.2590	0.4526	2.2093
2.64	0.5005	7.9645	3.4937	2.2797	0.4452	2.2461
2.66	0.4988	8.0882	3.5157	2.3006	0.4379	2.2835
2.68	0.4972	8.2128	3.5374	2.3217	0.4307	2.3218
2.7	0.4956	8.3383	3.5590	2.3429	0.4236	2.3608
2.72	0.4941	8.4648	3.5803	2.3642	0.4166	2.4005
2.74	0.4926	8.5922	3.6015	2.3858	0.4097	2.4411
2.76	0.4911	8.7205	3.6224	2.4074	0.4028	2.4825
2.78	0.4896	8.8498	3.6431	2.4292	0.3961	2.5246
2.8	0.4882	8.9800	3.6636	2.4512	0.3895	2.5676
2.82	0.4868	9.1111	3.6838	2.4733	0.3829	2.6115
2.84	0.4854	9.2432	3.7039	2.4955	0.3765	2.6561
2.86	0.4840	9.3762	3.7238	2.5179	0.3701	2.7017
2.88	0.4827	9.5101	3.7434	2.5405	0.3639	2.7481
2.9	0.4814	9.6450	3.7629	2.5632	0.3577	2.7954
2.92	0.4801	9.7808	3.7821	2.5861	0.3517	2.8436
2.94	0.4788	9.9175	3.8012	2.6091	0.3457	2.8927
2.96	0.4776	10.0552	3.8200	2.6322	0.3398	2.9427
2.98	0.4764	10.1938	3.8387	2.6555	0.3340	2.9937
3.0	0.4752	10.3333	3.8571	2.6790	0.3283	3.0456
3.02	0.4740	10.4738	3.8754	2.7026	0.3227	3.0985
3.04	0.4729	10.6152	3.8935	2.7264	0.3172	3.1523
3.06	0.4717	10.7575	3.9114	2.7503	0.3118	3.2072
3.08	0.4706	10.9008	3.9291	2.7744	0.3065	3.2630
3.1	0.4695	11.0450	3.9466	2.7986	0.3012	3.3199
3.12	0.4685	11.1901	3.9639	2.8230	0.2960	3.3778
3.14	0.4674	11.3362	3.9811	2.8475	0.2910	3.4368
3.16	0.4664	11.4832	3.9981	2.8722	0.2860	3.4969
3.18	0.4654	11.6311	4.0149	2.8970	0.2811	3.5580
3.2	0.4643	11.7800	4.0315	2.9220	0.2762	3.6202
3.22	0.4634	11.9298	4.0479	2.9471	0.2715	3.6835
3.24	0.4624	12.0805	4.0642	2.9724	0.2668	3.7480
3.26	0.4614	12.2322	4.0803	2.9979	0.2622	3.8136
3.28	0.4605	12.3848	4.0963	3.0234	0.2577	3.8803
3.3	0.4596	12.5383	4.1120	3.0492	0.2533	3.9483
3.32	0.4587	12.6928	4.1276	3.0751	0.2489	4.0174
3.34	0.4578	12.8482	4.1431	3.1011	0.2446	4.0877
3.36	0.4569	13.0045	4.1583	3.1273	0.2404	4.1593

Continuation of the Table B.2

1	2	3	4	5	6	7
3.38	0.4560	13.1618	4.1734	3.1537	0.2363	4.2321
3.4	0.4552	13.3200	4.1884	3.1802	0.2322	4.3062
3.42	0.4544	13.4791	4.2032	3.2069	0.2282	4.3815
3.44	0.4535	13.6392	4.2178	3.2337	0.2243	4.4581
3.46	0.4527	13.8002	4.2323	3.2607	0.2205	4.5361
3.48	0.4519	13.9621	4.2467	3.2878	0.2167	4.6154
3.5	0.4512	14.1250	4.2609	3.3151	0.2129	4.6960
3.52	0.4504	14.2888	4.2749	3.3425	0.2093	4.7780
3.54	0.4496	14.4535	4.2888	3.3701	0.2057	4.8614
3.56	0.4489	14.6192	4.3026	3.3978	0.2022	4.9461
3.58	0.4481	14.7858	4.3162	3.4257	0.1987	5.0324
3.6	0.4474	14.9533	4.3296	3.4537	0.1953	5.1200
3.62	0.4467	15.1218	4.3429	3.4819	0.1920	5.2091
3.64	0.4460	15.2912	4.3561	3.5103	0.1887	5.2997
3.66	0.4453	15.4615	4.3692	3.5388	0.1855	5.3918
3.68	0.4446	15.6328	4.3821	3.5674	0.1823	5.4854
3.7	0.4439	15.8050	4.3949	3.5962	0.1792	5.5806
3.72	0.4433	15.9781	4.4075	3.6252	0.1761	5.6773
3.74	0.4426	16.1522	4.4200	3.6543	0.1731	5.7756
3.76	0.4420	16.3272	4.4324	3.6836	0.1702	5.8755
3.78	0.4414	16.5031	4.4447	3.7130	0.1673	5.9770
3.8	0.4407	16.6800	4.4568	3.7426	0.1645	6.0801
3.82	0.4401	16.8578	4.4688	3.7723	0.1617	6.1849
3.84	0.4395	17.0365	4.4807	3.8022	0.1589	6.2915
3.86	0.4389	17.2162	4.4924	3.8323	0.1563	6.3997
3.88	0.4383	17.3968	4.5041	3.8625	0.1536	6.5096
3.9	0.4377	17.5783	4.4156	3.8928	0.1510	6.6213
3.92	0.4372	17.7608	4.5270	3.9233	0.1485	6.7348
3.94	0.4366	17.9442	4.5383	3.9540	0.1460	6.8501
3.96	0.4360	18.1285	4.5494	3.9848	0.1435	6.9672
3.98	0.4355	18.3138	4.5605	4.0158	0.1411	7.0861
4.0	0.4350	18.5000	4.5714	4.0469	0.1388	7.2069
4.02	0.4344	18.6871	4.5823	4.0781	0.1364	7.3296
4.04	0.4339	18.8752	4.5930	4.1096	0.1342	7.4542
4.06	0.4334	19.0642	4.6036	4.1412	0.1319	7.5807
4.08	0.4329	19.2541	4.6141	4.1729	0.1297	7.7092
4.1	0.4324	19.4450	4.6245	4.2048	0.1276	7.8397
4.12	0.4319	19.6368	4.6348	4.2368	0.1254	7.9722
4.14	0.4314	19.8295	4.6450	4.2690	0.1234	8.1067
4.16	0.4309	20.0232	4.6550	4.3014	0.1213	8.2433

Continuation of the Table B.2

1	2	3	4	5	6	7
4.18	0.4304	20.2178	4.6650	4.3339	0.1193	8.3819
4.2	0.4299	20.4133	4.6749	4.3666	0.1173	8.5227
4.22	0.4295	20.6098	4.6847	4.3994	0.1154	8.6656
4.24	0.4290	20.8072	4.6944	4.4324	0.1135	8.8107
4.26	0.4286	21.0055	4.7040	4.4655	0.1116	8.9579
4.28	0.4281	21.2048	4.7135	4.4988	0.1098	9.1074
4.3	0.4277	21.4050	4.7229	4.5322	0.1080	9.2591
4.32	0.4272	21.6061	4.7322	4.5658	0.1062	9.4131
4.34	0.4268	21.8082	4.7414	4.5995	0.1045	9.5694
4.36	0.4264	22.0112	4.7505	4.6334	0.1028	9.7280
4.38	0.4260	22.2151	4.7595	4.6675	0.1011	9.8889
4.4	0.4255	22.4200	4.7685	4.7017	0.0995	10.0522
4.42	0.4251	22.6258	4.7773	4.7361	0.0979	10.2179
4.44	0.4247	22.8325	4.7861	4.7706	0.0963	10.3861
4.46	0.4243	23.0402	4.7948	4.8053	0.0947	10.5567
4.48	0.4239	23.2488	4.8034	4.8401	0.0932	10.7298
4.5	0.4236	23.4583	4.8119	4.8751	0.0917	10.9054
4.52	0.4232	23.6688	4.8203	4.9102	0.0902	11.0835
4.54	0.4228	23.8802	4.8287	4.9455	0.0888	11.2643
4.56	0.4224	24.0925	4.8369	4.9810	0.0874	11.4476
4.58	0.4220	24.3058	4.8451	5.0166	0.0860	11.6336
4.6	0.4217	24.5200	4.8532	5.0523	0.0846	11.8222
4.62	0.4213	24.7351	4.8612	5.0882	0.0832	12.0136
4.64	0.4210	24.9512	4.8692	5.1243	0.0819	12.2076
4.66	0.4206	25.1682	4.8771	5.1605	0.0806	12.4044
4.68	0.4203	25.3861	4.8849	5.1969	0.0793	12.6040
4.7	0.4199	25.6050	4.8926	5.2334	0.0781	12.8065
4.72	0.4196	25.8248	4.9002	5.2701	0.0769	13.0117
4.74	0.4192	26.0455	4.9078	5.3070	0.0756	13.2199
4.76	0.4189	26.2672	4.9153	5.3440	0.0745	13.4210
4.78	0.4186	26.4898	4.9227	5.3811	0.0733	13.6450
4.8	0.4183	26.7133	4.9301	5.4184	0.0721	13.8620
4.82	0.4179	26.9378	4.9374	5.4559	0.0710	14.0820
4.84	0.4176	27.1632	4.9446	5.4935	0.0699	14.3050
4.86	0.4173	27.3895	4.9518	5.5313	0.0688	14.5312
4.88	0.4170	27.6168	4.9589	5.5692	0.0677	14.7604
4.9	0.4167	27.8450	4.9659	5.6073	0.0667	14.9928
4.92	0.4164	28.0741	4.9728	5.6455	0.0657	15.2284
4.94	0.4161	28.3042	4.9797	5.6839	0.0647	15.4672
4.96	0.4158	28.5352	4.9865	5.7224	0.0637	15.7902

Continuation of the Table B.2

1	2	3	4	5	6	7
4.98	0.4155	28.7671	4.9933	5.7611	0.0327	15.9545
5.0	0.4152	29.0000	5.0000	5.8000	0.0617	16.2032